ON THE BOUNDARY VALUE PROBLEMS OF LINEAR ORDINARY DIFFERENTIAL EQUATIONS OF SECOND ORDER*

BY

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It is the purpose of this paper to investigate the solutions of linear differential equations of second order which satisfy linear boundary conditions of a general form. In the first section theorems regarding the existence of solutions of the differential equation

$$y'' + p(x)y' + q(x)y = f(x)$$

under the boundary conditions

$$a_1 y(x_1) + a_2 y(x_2) + a_3 y'(x_1) + a_4 y'(x_2) = A,$$
$$b_1 y(x_1) + b_2 y(x_2) + b_3 y'(x_1) + b_4 y'(x_2) = B,$$

will be established, under the assumption that the relation

$$a_2 b_4 - a_4 b_2 = e^{\int_{x_1}^{x_2} p dx}(a_1 b_3 - a_3 b_1)$$

is satisfied. † These results will be derived from the general solution of the differential equation. In the following sections the differential equation

$$y'' + p(x)y' + [\lambda A(x) - B(x)]y = 0$$

will be considered, where λ is a parameter. The existence of functions satisfying the differential equation and the homogeneous form of the above boundary conditions for particular values of λ, i. e., the existence of normal functions for the problem, will be proved by means of certain minimum problems, based upon that used by WEBER ‡ whereby reasoning similar to that of DIRICHLET's prin-

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† The problem might be called in this case a self adjoint boundary problem. The nature of the problem would be considerably different if this relation were not satisfied.

‡ Mathematische Annalen, vol. 1 (1869), p. 1.

ciple was employed. By this method the properties of the normal functions as minimum functions are set in evidence, properties upon which the proof of oscillation theorems for the normal functions may be based. As an illustration an oscillation theorem for periodic solutions will be proved in the last section.

Special cases of the above boundary conditions have formed the subject of many investigations since the famous memoirs of STURM and LIOUVILLE.* PICARD† has applied the method of successive approximations to the case $y(x_1) = 0$, $y(x_2) = 0$, for the differential equation

$$y'' + \lambda A(x) y = 0,$$

where the function A does not change sign. The writer‡ has treated certain boundary conditions for this differential equation by reducing the problem to the solution of an integral equation by means of one-dimensional Green's functions, and applying a method to which the method of § 2 of the present paper is analogous. In particular the existence of periodic solutions was thus shown. These results were later reproduced by HOLMGREN,§ using a method based upon HILBERT's treatment of DIRICHLET's principle, and by TZITZÉICA,‖ who used a generalization of PICARD's method. The existence of periodic solutions was also proved by BÔCHER,¶ the proof being based on STURM's results.

Important results have recently been obtained by HILBERT regarding the solution of integral equations with symmetrical *Kern*.** These results may however be applied to the boundary problem of the differential equation only in case the function $A(x)$ does not change sign, a restriction which will not be made in the present article. The expansion of a function in terms of normal functions, which is the result of greatest interest in HILBERT's work, has also been treated by DIXON†† for a general boundary condition in connection with a differential equation to which the one here considered may be reduced if $A(x)$ does not change sign.

§ 1. *The existence of solutions of the general problem.*

It is required to determine a solution of the differential equation ‡‡

*Journal de Mathématiques, vols. 1, 2 (1836, 1837). See BÔCHER, *Encyklopädie der mathematischen Wissenschaften*, II A 7a.

†See his *Traité d'analyse*, 3 (1896), p. 105 ff.

‡Mathematische Annalen, vol. 58 (1904), p. 528.

§Arkiv för Matematik, Astronomi och Fysik, vol. 1 (1904), p. 401.

‖Comptes rendus, vol. 140 (1905), pp. 223, 492.

¶Ibid., vol. 140 (1905), p. 928.

**Nachrichten der K. Gesellschaft der Wissenschaften zu Göttingen, 1904. See also WESTFALL, Dissertation, Göttingen, 1905, and SCHMIDT, Dissertation, Göttingen, 1905.

††Proceedings of the London Mathematical Society, ser. 2, vol. 3 (1905), p. 83.

‡‡The coefficients of the different equations considered are assumed to be continuous functions in the interval (x_1, x_2).

$$y'' + p(x)y' + q(x)y = f(x) \tag{1}$$

which satisfies the boundary conditions

$$\begin{cases} L_a(y) \equiv a_1 y(x_1) + a_2 y(x_2) + a_3 y'(x_1) + a_4 y'(x_2) = A, \\ L_b(y) \equiv b_1 y(x_1) + b_2 y(x_2) + b_3 y'(x_1) + b_4 y'(x_2) = B, \end{cases} \tag{2}$$

where A, B, a_i, b_i are given constants. It will be assumed that the coefficients of the boundary conditions satisfy the relation

$$a_2 b_4 - a_4 b_2 = e^{\int_{x_1}^{x_2} p dx}(a_1 b_3 - a_3 b_1),$$

or, in writing

$$a_i b_k - a_k b_i = d_{ik}, \qquad e^{\int_{x_1}^{x_2} p dx} = \pi,$$

that the equation holds:

$$d_{24} = \pi d_{13}. \tag{3}$$

The general solution of equation (1) is

$$y = c_1 u + c_2 v + u\int_{x_1}^{x} fPv dx + v\int_{x}^{x_2} fPu dx, \tag{4}$$

where

$$P = e^{\int_{x_1}^{x} p dx},$$

c_1 and c_2 are arbitrary constants, and u, v are linearly independent solutions of the differential equation

$$y'' + p(x)y' + q(x)y = 0, \tag{1$_0$}$$

and are connected by the relation

$$u'v - v'u = e^{-\int_{x_1}^{x} p dx} = \frac{1}{P}. \tag{5}$$

The constants c_1, c_2 are to be so determined that y satisfies the boundary conditions (2). Substituting (4) in (2), writing $v(x_1) = v_1$, $v'(x_1) = v_1'$, etc., and arranging terms, we have:

$$\begin{cases} c_1 L_a(u) + c_2 L_a(v) = A - (a_1 v_1 + a_3 v_1')\int_{x_1}^{x_2} fPu dx - (a_2 u_2 + a_4 u_2')\int_{x_1}^{x_2} fPv dx, \\ c_1 L_b(u) + c_2 L_b(v) = B - (b_1 v_1 + b_3 v_1')\int_{x_1}^{x_2} fPu dx - (b_2 u_2 + b_4 u_2')\int_{x_1}^{x_2} fPv dx. \end{cases} \tag{6}$$

If in place of u, v any other pair of linearly independent solutions

$$U = d_1 u + e_1 v, \qquad V = d_2 u + e_2 v \qquad (d_1 e_2 - e_1 d_2 \neq 0)$$

is used, the determinant $\Delta(u, v)$ of the first members of (6) will be replaced by

$$\Delta(U, V) = (d_1 e_2 - e_1 d_2)\Delta(u, v).$$

Then the following theorem holds;

If $\Delta \neq 0$ there exists a unique solution of the differential equation (1) *under the boundary condition* (2). *There exists a solution of the homogeneous problem, i. e., a solution not identically zero of the differential equation* (1_0) *under the boundary conditions*

$$(2_0) \qquad \begin{cases} L_a(y) = 0, \\ L_b(y) = 0, \end{cases}$$

when and only when $\Delta = 0$.

Certain conditions must be satisfied by A, B, $f(x)$ in order that a solution of the non-homogeneous problem may exist when $\Delta = 0$. These conditions will be different according as the minors of Δ are or are not zero. The discussion of these two cases will be given separately.

Case I. The minors of Δ are not all zero.

In this case but one solution of the homogeneous problem exists, except for an arbitrary constant factor. Let u be this solution. Then v will not be a solution, i. e., $L_a(v)$ and $L_b(v)$ are not both zero. Since u satisfies (2_0) it will also satisfy the equations:

$$(7_1) \quad b_1 L_a(u) - a_1 L_b(u) \equiv d_{21} u(x_2) + d_{31} u'(x_1) + d_{41} u'(x_2) = 0,$$

$$(7_2) \quad b_2 L_a(u) - a_2 L_b(u) \equiv d_{12} u(x_1) + d_{32} u'(x_1) + \pi d_{31} u'(x_2) = 0,$$

$$(7_3) \quad b_3 L_a(u) - a_3 L_b(u) \equiv d_{13} u(x_1) + d_{23} u(x_2) + d_{43} u'(x_2) = 0,$$

$$(7_4) \quad b_4 L_a(u) - b_4 L_b(u) \equiv d_{14} u(x_1) + \pi d_{13} u(x_2) + d_{34} u'(x_1) = 0.$$

The necessary and sufficient condition for the existence of a solution of (1) and (2) is from (6):

$$L_a(v)\left\{ B - (b_1 v_1 + b_3 v_1')\int_{x_1}^{x_2} fPu\,dx - (b_2 u_2 + b_4 u_2')\int_{x_1}^{x_2} fPv\,dx \right\}$$

$$- L_b(v)\left\{ A - (a_1 v_1 + a_3 v_1')\int_{x_1}^{x_2} fPu\,dx - (a_2 u_2 + a_4 u_2')\int_{x_1}^{x_2} fPv\,dx \right\} = 0.$$

Since u satisfies the boundary condition (2_0) the coefficient of the integral involving v takes the form:

$$(a_1v_1 + a_3v_1')(b_1u_1 + b_3u_1') - (a_2v_2 + a_4v_2')(b_2u_2 + b_4u_2')$$
$$- (b_1v_1 + b_3v_1')(a_1u_1 + a_3u_1') + (b_2v_2 + b_4v_2')(a_2u_2 + a_4u_2')$$
$$= d_{13}(v_1u_1' - u_1v_1') - d_{24}(v_2u_2' - u_2v_2').$$

From (3) and (5) this reduces to

$$d_{13} - d_{24}e^{-\int_{x_1}^{x_2} p dx} = d_{13} - \frac{d_{24}}{\pi} = 0,$$

and the above condition becomes:

$$(8)\quad L_a(v)\left\{ B - (b_1v_1 + b_3v_1')\int_{x_1}^{x_2} fPudx \right\}$$
$$- L_b(v)\left\{ A - (a_1v_1 + a_3v_1')\int_{x_1}^{x_2} fPudx \right\} = 0.$$

Now it may be shown by means of equations (3), (5) and (7) that the following equations hold:

$$(9_1)\qquad L_a(v)(b_2u_1' + \pi b_1u_2') - L_b(v)(a_2u_1' + \pi a_1u_2') = 0,$$

$$(9_2)\qquad L_a(v)(b_4u_1 + \pi b_3u_2) - L_b(v)(a_4u_1 - \pi a_3u_2) = 0,$$

$$(9_3)\qquad L_a(v)(\pi b_3u_2' - b_2u_1) - L_b(v)(\pi a_3u_2' - a_2u_1) = 0,$$

$$(9_4)\qquad L_a(v)(b_4u_1' - \pi b_1u_2) - L_b(v)(a_4u_1' - \pi a_1u_2) = 0.$$

In fact, expanding the left member of (9_1) and applying (3), (5) and (7), we have,

$$d_{12}v_1u_1' + \pi d_{21}v_2u_2' + d_{32}u_1'v_1' + \pi d_{31}v_1'u_2' + d_{42}u_1'v_2' + \pi d_{41}u_2'v_2'$$
$$= d_{12} + v_1'(d_{12}u_1 + d_{32}u_1' + \pi d_{31}u_2') + d_{21}$$
$$+ \pi v_2'(d_{21}u_2 + d_{31}u_1' + d_{41}u_2') = 0.$$

The other equations of (9) are proved in a similar manner. Now $L_a(v)$ and $L_b(v)$ are not both zero, and hence the equation

$$(10)\quad (a_2u_1' + \pi a_1u_2')\left\{ B - (b_1v_1 + b_3v_1')\int_{x_1}^{x_2} fPudx \right\}$$
$$- (b_2u_1' + \pi b_1u_2')\left\{ A - (a_1v_1' + a_3v_1')\int_{x_1}^{x_2} fPudx \right\} = 0$$

is a result of (8) and (9_1). Furthermore, (8) is a result of (9_1) and (10) unless

$$a_2u_1' + \pi a_1u_2' = 0, \qquad b_2u_1' + \pi b_1u_2' = 0.$$

These equations can hold only when either $d_{12}=0$ or $u_1'=0$, $u_2'=0$. But if $u_1'=0$, $u_2'=0$ then $u_1\neq 0$, $u_2\neq 0$, for a solution of (1_0) can not vanish together with its derivative, and it follows from (7_1) that $d_{12}=0$. Therefore equation (10) is equivalent to (8) unless $d_{12}=0$. If we expand the terms of (10) we have

$$(a_2B-Ab_2)u_1'+\pi(a_1B-Ab_1)u_2' \\ -(d_{21}v_1u_1'+d_{23}v_1'u_1'+\pi d_{13}v_1'u_2')\int_{x_1}^{x_2}fPudx=0,$$

or, on account of (5) and (7_2),

$$(a_2B-Ab_2)u_1'+\pi(a_1B-Ab_1)u_2'+d_{12}\int_{x_1}^{x_2}fPudx=0,$$

an equation which does not contain v, and which is equivalent to (8) except when $d_{12}=0$. Three similar equations may be derived from (8) by using (9_2), (9_3) and (9_4). We have, replacing π and P by their values, the four equations,

$$(11)\begin{cases}(a_2B-Ab_2)u_1'+e^{\int_{x_1}^{x_2}pdx}(a_1B-Ab_1)u_2'+d_{12}\displaystyle\int_{x_1}^{x_2}fe^{\int_{x_1}^{x}pdx}udx=0,\\ (a_4B-Ab_4)u_1+e^{\int_{x_1}^{x_2}pdx}(a_3B-Ab_3)u_2+d_{43}\displaystyle\int_{x_1}^{x_2}fe^{\int_{x_1}^{x}pdx}udx=0,\\ (a_2B-Ab_2)u_1-e^{\int_{x_1}^{x_2}pdx}(a_3B-Ab_3)u_2'+d_{23}\displaystyle\int_{x_1}^{x_2}fe^{\int_{x_1}^{x}pdx}udx=0\\ (a_4B-Ab_4)u_1'-e^{\int_{x_1}^{x_2}pdx}(a_1B-Ab_1)u_2+d_{14}\displaystyle\int_{x_1}^{x_2}fe^{\int_{x_1}^{x}pdx}udx=0,\end{cases}$$

each of which is equivalent to (8) unless the determinant d_{ik} which occurs in it is zero. Since from equation (3) d_{13} and d_{24} vanish together it is easily seen that all the determinants d_{12}, d_{43}, d_{23}, d_{14} cannot be zero unless all determinants d_{ik} are zero, a case to be excluded, since the boundary conditions (2) would be either incompatible or dependent. Therefore:

The necessary and sufficient condition for the existence of a solution of the differential equation

$$(1)\qquad y''+py'+qy=f,$$

under the boundary conditions

$$(2)\qquad\begin{cases}L_a(y)\equiv a_1y(x_1)+a_2y(x_2)+a_3y'(x_1)+a_4y'(x_2)=A,\\ L_b(y)\equiv b_1y(x_1)+b_2y(x_2)+b_3y'(x_1)+b_4y'(x_2)=B,\end{cases}$$

when one solution, u, of the homogeneous problem

$$y'' + py' + qy = 0, \qquad L_a(y) = 0, \qquad L_b(y) = 0,$$

exists, but no other solution linearly independent of u, *is that* A, B, *and* $f(x)$ *satisfy one of the equations* (11) *with non-vanishing determinant* d_{ik}. *If* $A=0$, $B=0$ *the condition is*

$$\int_{x_1}^{x_2} f e^{\int_{x_1}^{x} p dx}\, u dx = 0. \tag{12}$$

If this condition is satisfied there exists a single infinity of solutions of (1) *and* (2) *of the form*

$$y = \bar{y} + cu,$$

where $\bar{y}$ *is any one such solution and* c *is an arbitrary constant.*

CASE II. *All minors of* Δ *are zero.*

In this case the functions u and v are solutions of the homogeneous problem. The determinant d_{13} cannot be zero. For if $d_{13}=0$ then $d_{24}=0$ and since v satisfies the same boundary conditions as u it follows from (7_1) that

$$d_{21}u(x_2) + d_{41}u'(x_2) = 0,$$

$$d_{21}v(x_2) + d_{41}v'(x_2) = 0,$$

and therefore $d_{12}=0$, $d_{14}=0$, for the determinant $u(x_2)v'(x_2) - v(x_2)u'(x_2)$ is not zero. From equation (7_3) it follows in the same way that $d_{23}=0$, $d_{34}=0$ in case $d_{13}=0$, that is, all determinants d_{ik} are zero if d_{13} is zero. We may therefore assume that d_{13} is not zero.

The necessary and sufficient conditions for the existence of a solution of equations (1) and (2) are seen from (6) and the boundary conditions for u and v to be

$$A - (a_1v_1 + a_3v_1')\int_{x_1}^{x_2} fPu dx + (a_1u + a_3u_1')\int_{x_1}^{x_2} fPv dx = 0,$$

$$B - (b_1v_1 + b_3v_1')\int_{x_1}^{x_2} fPu dx + (b_1u + b_3u')\int_{x_1}^{x_2} fPv dx = 0,$$

Eliminating each integral in turn we have, on account of (5),

$$\begin{cases} (a_1B - Ab_1)u_1 + (a_3B - Ab_3)u_1' + d_{13}\displaystyle\int_{x_1}^{x_2} f e^{\int_{x_1}^{x} p dx}\, u dx = 0, \\ (a_1B - Ab_1)v_1 + (a_3B - Ab_3)v_1' + d_{13}\displaystyle\int_{x_1}^{x_2} f e^{\int_{x_1}^{x} p dx}\, v dx = 0. \end{cases} \tag{13}$$

Since d_{13} can not be zero in this case, these equations are equivalent to the preceding pair, and hence:

The equations (13) *are the necessary and sufficient conditions for the existence of a solution of the differential equation* (1) *under the boundary condition* (2) *in case two linearly independent solutions* u, v *of the homogeneous problem exist. If* $A = 0$, $B = 0$ *the conditions are*

$$\int_{x_1}^{x_2} f e^{\int_{x_1}^{x_2} p dx} u dx = 0, \qquad \int_{x_1}^{x_2} f e^{\int_{x_1}^{x_2} p dx} v dx = 0. \tag{14}$$

If these conditions are satisfied there exists a doubly infinite set of solutions of (1) *and* (2) *of the form*

$$y = \bar{y} + cu + dv,$$

where $\bar{y}$ *is any one such solution and* c, d *are arbitrary constants.*

§ 2. *The existence of the first normal functions.*

There exists a solution of the homogeneous problem when and only when the determinant Δ of equations (6) is zero. If the coefficients of the differential equation contain a parameter λ, then Δ is a function of λ, and λ must have a value which is a zero of $\Delta(\lambda)$ in order that a solution of the required type exists. These zeros λ_i of $\Delta(\lambda)$ will be called *normal parameter values*, and the corresponding solutions y_i of the homogeneous problem *normal functions*. We shall consider the case that q is a linear function of λ. The problem to be treated is to prove the existence and investigate the properties of the normal parameter values and normal functions for the differential equation

$$y'' + p(x)y' + [\lambda A(x) - B(x)]\, y = 0 \tag{1'}$$

under the boundary conditions

$$\begin{cases} L_a(y) \equiv a_1 y(x_1) + a_2 y(x_2) + a_3 y'(x_1) + a_4 y'(x_2) = 0, \\ L_b(y) \equiv b_1 y(x_1) + b_2 y(x_2) + b_3 y'(x_1) + b_4 y'(x_2) = 0. \end{cases} \tag{2_0}$$

It will be assumed that the following conditions are satisfied:

$$d_{24} = \pi d_{13} \qquad \left(\pi = e^{\int_{x_1}^{x_2} p dx}\right), \tag{3}$$

$$B(x) \geqq 0 \qquad (x_1 \leqq x \leqq x_2), \tag{15}$$

(16) *all the determinants* d_{12}, d_{14}, d_{23}, d_{43} *that are not zero have the same sign.*

The following statement is a result of condition (16):

LEMMA. *The values of the expression*

$$-\left[yy' e^{\int_{x_1}^{x} p dx}\right]_{x_1}^{x_2} \equiv y(x_1)y'(x_1) - \pi y(x_2)y'(x_2)$$

formed for all functions which satisfy (2_0) *are positive or zero.*

Any function y which satisfies (2_0) also satisfies the equations

$$(17_1) \qquad d_{21}y(x_2) + d_{31}y'(x_1) + d_{41}y'(x_2) = 0,$$

$$(17_2) \qquad d_{12}y(x_1) + d_{32}y'(x_1) + \pi d_{31}y'(x_2) = 0,$$

$$(17_3) \qquad d_{13}y(x_1) + d_{23}y(x_2) + d_{43}y'(x_2) = 0,$$

$$(17_4) \qquad d_{14}y(x_1) + \pi d_{13}y(x_2) + d_{34}y'(x_1) = 0.$$

Hence

$$(18_1)\ d_{12}[y(x_1)y'(x_1) - \pi y(x_2)y'(x_2)] = d_{23}[y'(x_1)]^2 + 2\pi d_{13}y'(x_1)y'(x_2) + \pi d_{14}[y'(x_2)]^2,$$

$$(18_2)\ d_{43}[y(x_1)y'(x_1) - \pi y(x_2)y'(x_2)] = d_{14}[y(x_1)]^2 + 2\pi d_{13}y(x_1)y(x_2) + \pi d_{23}[y(x_2)]^2,$$

$$(18_3)\ d_{23}[y(x_1)y'(x_1) - \pi y(x_2)y'(x_2)] = d_{12}[y(x_1)]^2 + \pi d_{43}[y'(x_2)]^2,$$

$$(18_4)\ d_{14}[y(x_1)y'(x_1) - \pi y(x_2)y'(x_2)] = d_{43}[y'(x_1)]^2 + \pi d_{12}[y(x_2)]^2.$$

The discriminants of the right members of (18_1) and (18_2) are

$$\pi(d_{14}d_{23} - \pi d_{13}^2) = \pi(d_{14}d_{23} - d_{13}d_{24}).$$

The second member when expanded is seen to be equal to

$$\pi d_{12}d_{43}.$$

Since the determinants d_{ik} which occur in the first members of (18) cannot all be zero the truth of the above lemma is seen at once.

Consider now the following minimum problem:

Among all functions $y(x)$ which satisfy the boundary conditions (2_0) and the equation *

$$(19) \qquad \int_{x_1}^{x_2} APy^2\,dx = 1 \qquad \left(P = e^{\int_{x_1}^{x} p\,dx}\right),$$

and are continuous together with their first and second derivatives in the interval (x_1, x_2), that one is required which gives the least value to the expression

$$J = \int_{x_1}^{x_2} (y'^2 + By^2)\,P\,dx - [yy'P]_{x_1}^{x_2}.$$

The values of J are all positive or zero, on account of conditions (15) and the lemma proved by means of condition (16). Hence there exists a definite *lower limit* λ_0 of the values of J under the above conditions. It will be proved that

* It is here assumed that A is not everywhere negative in (x_1, x_2). This is no restriction on the problem since A occurs in the equation (1′) multiplied by the parameter λ.

this value λ_0 is a normal parameter value for the differential equation (1′) under the boundary conditions (2_0), and that the corresponding normal function is the solution of the minimum problem. If λ_0 were not a normal parameter value there would exist, according to the theorems of § 1, a solution of the differential equation

$$y'' + py' + (\lambda_0 A - B)y = f$$

under the boundary conditions (2_0), whatever be f. It will be shown that f could be so chosen that the solution of the equation would satisfy all conditions of the minimum problem, and would give to J a value less than the lower limit of all such values, λ_0. The value λ_0 must therefore be a normal parameter value.

Let

$$u_1, u_2, u_3, \cdots$$

be an infinite series of approximating functions for the minimum problem, i. e., an infinite series of functions u_h which satisfy the equation

$$\lim_{h=\infty} J(u_h) = \lambda_0, \tag{20}$$

and all the conditions of the minimum problem.

LEMMA. *The approximating functions u_h can be so chosen that there exists a fixed number G such that for all h*

$$|u_h(x)| < G \qquad (x_1 \leqq x \leqq x_2) \tag{21}$$

From the inequalities,

$$\left| \int_{x_1}^{x} u_h' \, dx \right| \leqq \int_{x_1}^{x} |u_h'| \, dx < \int_{x_1}^{x} (u_h'^2 + 1) \, dx,$$

$$\int_{x_1}^{x} u_h' \, dx \leqq \frac{1}{m} \int_{x_1}^{x} P u_h'^2 dx < \frac{J(u_h)}{m},$$

where m is the minimum of $P = e^{\int_{x_1}^{x} p dx}$ in the interval (x_1, x_2), it follows that

$$|u_h(x) - u_h(x_1)| < \frac{\lambda_0}{m} + (x_2 - x_1) + \epsilon_h,$$

where $\lim_{h=\infty} \epsilon_h = 0$. The functions u_h therefore have the form

$$u_h(x) = c_h + \gamma_h(x),$$

where c_h are constants and γ_h a series of functions such that the maxima of their absolute values remain under a fixed limit. From the equation

$$\int_{x_1}^{x_2} AP u_h^2 dx = c_h^2 \int_{x_1}^{x_2} AP dx + 2c_h \int_{x_1}^{x_2} AP \gamma_h dx + \int_{x_1}^{x_2} AP \gamma_h^2 dx = 1,$$

it follows that the quantities c_h all lie between finite limits unless

$$\int_{x_1}^{x_2} AP dx = 0, \qquad \lim_{h=\infty} \int_{x_1}^{x_2} A \gamma_h dx = 0,$$

and consequently

$$\lim_{h=\infty} \int_{x_1}^{x_2} A \gamma_h^2 dx = 1. \tag{22}$$

From equation (18_4), which is satisfied by the functions u_h, we have

$$d_{43}[u_h'(x_1)]^2 + \pi d_{12}[u_h(x_2)]^2 = -d_{14}[u_h u_h' P]_{x_1}^{x_2} \leqq d_{12} J(u_h).$$

Hence, if $|c_h|$ increase beyond limit, $d_{12} = 0$ and either $d_{43} = 0$, or the values $u_h'(x_1)$ remain between finite limits. Similarly from (18_3), either $d_{43} = 0$, or $u_h'(x_2)$ remain between finite limits, and hence, from (17_3) and (17_4),

$$d_{13} + d_{23} = 0, \qquad d_{14} + \pi d_{13} = 0.$$

Therefore either $d_{13} \neq 0$ or $d_{34} \neq 0$, for if both are zero then all determinants d_{ik} are zero. Then the boundary conditions (2_0) are equivalent to either (17_1) and (17_3), or to (17_3) and (17_4). Hence, if $|c_h|$ increase beyond limit, the boundary conditions, since d_{12} must be zero, are equivalent to

$$y'(x_1) - \pi y'(x_2) = 0,$$

$$y(x_1) - y(x_2) = 0,$$

if $d_{34} = 0$, or to

$$d_{13}[y(x_1) - y(x_2)] + d_{43} y'(x_2) = 0,$$

$$\pi d_{13}[y(x_1) - y(x_2)] + d_{43} y'(x_1) = 0,$$

if $d_{34} \neq 0$. [In either case the boundary conditions are satisfied by $y = \text{const.}$, and hence by $\gamma_h = u_h - c_h$], and we have also

$$-[\gamma_h \gamma_h' P]_{x_1}^{x_2} = -[u_h u_h' P]_{x_1}^{x_2} + c_h[\pi u_h'(x_2) - u_h'(x_1)] = -[u_h u_h' P]_{x_1}^{x_2}.$$

Therefore

$$J(\gamma_h) = J(u_h).$$

Let

$$\delta_h = \int_{x_1}^{x_2} AP \gamma_h^2 dx.$$

From equation (22)

$$\lim_{h=\infty} \delta_h = 1.$$

Therefore if $|c_h|$ increase without limit, i. e., if the functions u_h do not satisfy condition (21) there exists a series of functions

$$\bar{u}_h(x) = \frac{u_h(x) - c_h}{\sqrt{|\delta_h|}}$$

which satisfy (21), and since they satisfy the equations

$$L_a(\bar{u}_h) = 0, \qquad L_b(\bar{u}_h) = 0,$$

$$\int_{x_1}^{x_2} AP\bar{u}_h^2\,dx = \frac{1}{\delta_h}\int_{x_1}^{x_2} AP\gamma_h^2\,dx = 1,$$

$$\lim_{h=\infty} J(\bar{u}_h) = \lim_{h=\infty} \frac{1}{\delta_h} J(u_h) = \lambda_0,$$

they form a series of approximating functions for the minimum problem. The lemma stated above is therefore proved.

Define an infinite series of functions f_h by the equations

$$(u_h' P)' + (\lambda_0 A - B)Pu_h = f_h. \tag{23}$$

Multiplying these equations by u_h and integrating from x_1 to x_2 we have, on account of the condition (19),

$$[u_h u_h' P]_{x_1}^{x_2} - \int_{x_1}^{x_2} (u_h'^2 + Bu_h^2)\,Pdx + \lambda_0 = \int_{x_1}^{x_2} f_h u_h\,dx.$$

Hence, on account of (20),

$$\lim_{h=\infty} \int_{x_1}^{x_2} f_h u_h\,dx = 0. \tag{24}$$

Under the assumption that λ_0 is not a normal parameter value there exists for each h a solution of the differential equation

$$U_h'' + pU_h' + (\lambda_0 A - B)U_h = Au_h,$$

or, multiplying by $P = e^{\int_{x_1}^{x} p\,dx}$, of the equation

$$(U_h' P)' + (\lambda_0 A - B)PU_h = PAu_h, \tag{25}$$

under the boundary conditions (2_0). Consider the functions

$$v_h = u_h + cU_h$$

where c is a constant. These functions satisfy the boundary conditions (2_0) and the equation

$$(v_h' P)' + (\lambda_0 A - B)Pv_h = f_h + cPAu_h.$$

Multiplying this equation by $v_h = u_h + cU_h$ and integrating from x_1 to x_2 the following equation results:

$$(26)\qquad \begin{aligned}[v_h v_h' P]_{x_1}^{x_2} - \int_{x_1}^{x_2} (v_h'^2 + Bv_h^2) P dx + \lambda_0 \int_{x_1}^{x_2} APv_h^2 dx \\ = \int_{x_1}^{x_2} f_h u_h dx + c\int_{x_1}^{x_2} (f_h U_h + PAu_h^2) dx + c^2 \int_{x_1}^{x_2} PAu_h U_h dx.\end{aligned}$$

Now from equations (23) and (25) it follows that

$$\begin{aligned}\int_{x_1}^{x_2} (f_h U_h - PAu_h^2) dx &= \int_{x_1}^{x_2} \{(u_h' P)' U_h - (U_h' P) u_h\} dx \\ &= [(u_h' U_h - U_h' u_h) P]_{x_1}^{x_2}.\end{aligned}$$

The functions u_h and U_h satisfy the boundary conditions (2_0) and hence equations (17). Since $P(x_1) = 1$, $P(x_2) = \pi$, it follows from (17_1) and (17_2) that

$$\begin{aligned}d_{12}[(u_h' U_h - U_h' u_h) P]_{x_1}^{x_2} &= \pi u_h'(x_2)[d_{31} U_h'(x_1) + d_{41} U_h'(x_2)] \\ &\quad - \pi U_h'(x_2)[d_{31} u_h'(x_1) + d_{41} u_h'(x_2)] - u_h'(x_1)[d_{23} U_h'(x_1) + \pi d_{13} U_h'(x_2)] \\ &\quad + U_h'(x_1)[d_{23} u_h'(x_1) + \pi d_{13} u_h'(x_2)] = 0.\end{aligned}$$

Hence

$$[(u_h' U_h - U_h' u_h) P]_{x_1}^{x_2} = 0$$

unless $d_{12} = 0$. By using other pairs of equations (17) it is seen that this equation must hold in any case, since all the determinants d_{ik} cannot be zero. Therefore

$$\int_{x_1}^{x_2} (f_h u_h - PAu_h^2) dx = 0.$$

Since

$$\int_{x_1}^{x_2} PAu_h^2 dx = 1,$$

equation (26) becomes

$$(27)\qquad \lambda_0 \int_{x_1}^{x_2} PAv_h^2 dx - J(v_h) = \int_{x_1}^{x} f_h u_h dx + 2c + c^2 \int_{x_1}^{x_2} PAu_h U_h dx.$$

Now from equation (4)

$$U_h = c_h u + d_h v + u\int_{x_1}^{x} PAu_h v dx + v\int_{x}^{x_2} PAu_h u dx,$$

where u, v are the linearly independent solutions of the equation

$$y'' + py' + (\lambda A - B) y = 0$$

and c_h, d_h are constants whose values are so determined that U_h satisfies the boundary conditions (2_0). Referring to equations (6) one sees that c_h and d_h are linear combinations of the two integrals

$$\int_{x_1}^{x_2} PAu_h u dx, \qquad \int_{x_1}^{x_2} PAu_h v dx,$$

the coefficients being constants independent of h. Therefore since the functions u_h satisfy condition (20) it follows that the functions U_h satisfy a similar condition, and that the quantities

$$\left| \int_{x_1}^{x_2} PAu_h U_h dx \right|,$$

which occur in (27) remain for all h under some fixed limit B. Then from (27)

$$\lambda_0 \int_{x_1}^{x_2} PAv_h^2 dx - J(v_h) > \int_{x_1}^{x_2} f_h v_h dx + 2c - c^2 B.$$

Let c be chosen as a positive number so small that

$$2c - c^2 B > 0.$$

Write $2c - c^2 B = \delta$. It follows from (24) that h may be taken so large that

$$\int_{x_1}^{x_2} f_h u_h dx < \delta.$$

Then for this value of h we have

$$\lambda_0 \int_{x_1}^{x_2} PAv_h^2 dx - J(v_h) > 0. \tag{28}$$

Therefore

$$\int_{x_1}^{x_2} PAv_h^2 dx > 0,$$

and

$$\alpha = \left\{ \int_{x_1}^{x_2} PAv_h^2 dx \right\}^{\frac{1}{2}}$$

is a real number. Let

$$y = \frac{v_h}{\alpha}.$$

This function satisfies all conditions of the minimum problem. It must therefore give to J a value less than or equal to λ_0, the lower limit of the values of J under the conditions of the problem. But dividing (28) by α^2 we have

$$\lambda_0 - \frac{1}{\alpha^2} J(v_h) = \lambda_0 - J(y) > 0,$$

which is impossible. The assumption that λ_0 is not a normal parameter value therefore leads to a contradiction. *Hence the lower limit λ_0 is a normal parameter value, and there exists a normal function y_0, not identically zero, which satisfies the differential equation*

$$y_0'' + py_0' + (\lambda_0 A - B)y_0 = 0 \tag{29}$$

and the boundary conditions (2_0).

Multiplying equation (29) by Py_0 and integrating from x_1 to x_2, we derive the equation

$$\lambda_0 \int_{x_1}^{x_2} PAy_0^2 dx = J(y_0). \tag{30}$$

Since λ_0 is positive or zero the integral in the first member is positive unless $J(y_0) = 0$, i. e., unless y_0 is a constant. Hence the arbitrary constant factor of y_0 can be so determined that

$$\int_{x_1}^{x_2} PAy_0^2 dx = 1, \tag{31}$$

unless $y_0 = \text{const.}$ Even if $y_0 = \text{const.}$ this determination may be made, unless

$$\int_{x_1}^{x_2} PA dx = 0; \tag{32}$$

for if the integral is not zero it may be assumed without restriction to be positive, as the function A occurs in the differential equation multiplied by the parameter λ. Now equation (32) cannot hold if $y_0 = \text{const.}$, since if (32) is satisfied λ_0 must be greater than zero. In fact, since the approximating functions u_h satisfy the equation

$$\int_{x_1}^{x_2} PAu_h^2 dx = 1,$$

it follows that if (32) holds, each function u_h must assume for some two values $x = x_h$, $x = \bar{x}_h$ two values $u_h(x_h)$, $u_h(\bar{x}_h)$ which differ by an amount equal to δ, a fixed positive number independent of h. Let Y_h represent the ordinate of the straight line connecting the two points $[x_h, u_h(x_h)]$ and $[\bar{x}_h, u_h(\bar{x}_h)]$. Then

$$|Y_h'| = \frac{\delta}{\bar{x}_h - x_h}, \qquad Y'' = 0.$$

Now

$$J(u_h) \geqq \int_{x_h}^{\bar{x}_h} u_h'^2 dx.$$

Write

$$\eta_h = Y_h - u_h.$$

Then since η_h vanishes at x_h and at $\bar{x}_h$, and $Y'' = 0$,

$$\int_{x_h}^{\bar{x}_h} u_h'^2\,dx = \int_{x_h}^{\bar{x}_h} (Y_h' - \eta_h')^2\,dx = \frac{\delta^2}{\bar{x}_h - x_h} + \int_{x_h}^{\bar{x}_h} \eta_h'^2\,dx.$$

Therefore, for all values of h,

$$J(u_h) > \frac{\delta^2}{x_2 - x_1},$$

and λ_0 cannot be zero.

It may therefore be assumed without restriction that the normal function y_0 satisfies equation (31). Hence from (30) y_0 gives to J the value λ_0. *The normal function y_0 is therefore the solution of the minimum problem.*

§ 3. *The existence of an infinite series of normal functions.*

We proceed by complete induction. Suppose that there exist n positive or zero normal parameter values

$$\lambda_0, \lambda_1, \cdots, \lambda_{n-1} \qquad (\lambda_{i+1} \geqq \lambda_i)$$

and n linearly independent normal functions

$$y_0, y_1, \cdots, y_{n-1}$$

satisfying the differential equations

$$y_i'' + py_i' + (\lambda_i A - B)y_i = 0$$

and the boundary conditions (2_0). At most two of the values λ_i can coincide, since the functions y_i are assumed to be linearly independent. Multiplying the differential equation by Py_i and integrating from x_1 to x_2, we have

$$\lambda_i \int_{x_1}^{x_2} PAy_i^2\,dx = J(y_i).$$

The arbitrary constant factor of y_i may therefore be so determined that

$$\int_{x_1}^{x_2} PAy_i^2\,dx = 1, \tag{33}$$

except possibly when $J(y_i) = 0$, since λ_i and $J(y_i)$ are positive or zero. If $J(y_i) = 0$ then y_i is a constant and $\lambda_i = 0$; hence $y_i = y_0$, and it has been shown above that y_0 satisfies (31). Furthermore, from the equations

$$(Py_i')' + (\lambda_i A - B)y_i = 0,$$
$$(Py_k')' + (\lambda_k A - B)y_k = 0,$$

follows

$$(\lambda_i - \lambda_k)\int_{x_1}^{x_2} PAy_i y_k dx = [\,P(y_k' y_i - y_i' y_k)]_{x_1}^{x_2}.$$

Since y_i and y_k satisfy the boundary conditions (2_0), the right member is zero,* and hence if $\lambda_i \neq \lambda_k$,

$$\int_{x_1}^{x_2} PAy_i y_k dx = 0. \tag{34}$$

If $\lambda_i = \lambda_k$ then the functions y_i and y_k are not uniquely determined. The function y_k has the form

$$y_k = Y_k + cy_i,$$

where c is an arbitrary constant. The value of c may be determined so that

$$\int_{x_1}^{x_2} PAy_i y_k dx = \int_{x_1}^{x_2} PAy_i Y_k dx + c = 0.$$

It may therefore be assumed without restriction that the n normal functions y_i satisfy the equations

$$\int_{x_1}^{x_2} PAy_i^2 dx = 1 \tag{33}$$

$$\int_{x_1}^{x_2} PAy_i y_k dx = 0 \qquad (i \neq k). \tag{34}$$

To prove the existence of a normal function y_n linearly independent of $y_0, y_1, \cdots, y_{n-1}$, consider the following minimum problem:

It is required to minimize the expression

$$J(u) = \int_{x_1}^{x_2} (y'^2 + By^2) P dx - [Pyy']_{x_1}^{x_2}$$

under the conditions

$$L_a(y) = 0, \qquad L_b(y) = 0,$$

$$\int_{x_1}^{x_2} PAy^2 dx = 1, \tag{35}$$

$$\int_{x_1}^{x_2} PAy_i y dx = 0 \qquad (i = 0, 1, 2, \cdots, n-1). \tag{36}$$

Under the conditions (15), (16) there exists a finite lower limit λ_n of the values of J, and an infinite series of approximating functions

$$u_1, u_2, u_3, \cdots$$

* The proof is the same as that given in the reduction of equation (26) to (27).

which satisfy all the conditions of the minimum problem and the equation

$$\lim_{h=\infty} J(u_h) = \lambda_n .$$

Define as before an infinite series of functions f_h, by the equations

$$(Pu_h')' + (\lambda_n A - B)Pu_h = f_h .$$

Under the assumption that for $\lambda = \lambda_n$ there exists no solution of $(1')$ and (2_0) which is linearly independent of $y_0, y_1, \cdots, y_{n-1}$, there must exist, for each value of h, a solution of the equation

$$U_h'' + pU_h' + (\lambda_n A - B)U_h = Au_h \tag{37}$$

under the boundary conditions (2_0). For by the theorems of § 1 the equations

$$\int_{x_1}^{x_2} Ay_i u_h\, dx = 0, \int_{x_1}^{x_2} Ay_k u_h\, dx = 0 \qquad (h=1, 2, 3, \cdots),$$

which are satisfied by all the functions u_h, are sufficient conditions for the existence of such solutions U_h in case $\lambda_n = \lambda_i = \lambda_k$; the first equation alone is the sufficient condition in case λ_n is equal to but one of the normal values λ_i; and there is no condition necessary in case λ_n is not a normal value. The possibility that λ_n is a normal parameter value different from $\lambda_0, \lambda_1, \cdots, \lambda_{n-1}$ is excluded by the assumption that for $\lambda = \lambda_n$ no solution of $(1')$ and (2_0) exists which is linearly independent of $y_0, y_1, \cdots, y_{n-1}$.

Since

$$\int_{x_1}^{x_2} PAy_i u_h\, dx = 0 \qquad \begin{pmatrix} i=0, 1, \cdots, n-1 \\ h=1, 2, 3, \cdots \end{pmatrix},$$

it follows from the equations

$$(Py_i')' + (\lambda_i A - B)Py_i = 0,$$

$$(PU_h')' + (\lambda_n A - B)PU_h = PAu_h,$$

that

$$(\lambda_n - \lambda_i)\int_{x_1}^{x_2} PAy_i U_h\, dx = [P(y_i U_h - U_h' y_i)]_{x_1}^{x_2}.$$

The right member is zero since y_i and U_h both satisfy the boundary conditions (2_0). Hence if $\lambda_n \neq \lambda_i$ the functions v_h satisfy the equations

$$\int_{x_1}^{x_2} Ay_i U_h\, dx = 0 .$$

If $\lambda_n = \lambda_i$ then the functions U_h are not uniquely determined by (37) and (2_0), but have the form

$$U_h = \bar{U}_h + c_n y_i,$$

where c_n are arbitrary constants. These constants may be so determined that

$$\int_{x_1}^{x_2} PAy_i U_h \, dx = \int_{x_1}^{x_2} PAy_i U_h \, dx + c_h = 0.$$

The functions U_h, and therefore the functions

$$v_h = u_h + cU_h,$$

where c is a constant, satisfy all the conditions of the minimum problem except

$$\int_{x_1}^{x_2} PAv_h^2 \, dx = 1.$$

The assumption, that for $\lambda = \lambda_n$ no normal function linearly independent of $y_0, y_1, \cdots, y_{n-1}$ exists, may now be shown to be false by the method used in the case of the first normal function. There exists $n+1$ normal functions provided n exist, and the existence of a first has been proved. Hence an infinite series of normal functions y_n exists, and an infinite series of corresponding normal parameter values λ_n which are defined as the lower limits of the expression $J(y)$ under the conditions (2_0), (35), (36).

As has been shown above, it may be assumed without restriction that these functions satisfy the equations

$$\int_{x_1}^{x_2} PAy_n y_m \, dx = 0 \qquad (n \neq m),$$

$$\int_{x_1}^{x_2} PAy_n^2 \, dx = 1.$$

The function y_n therefore satisfies the conditions of the minimum problem used to define λ_n. Multiplying the equation

$$(Py_n')' + (\lambda_n A - B) Py_n = 0$$

by y_n and integrating from x_1 to x_2, we have

$$J(y_n) = \lambda_n.$$

Hence y_n is the solution of the minimum problem. From the definition of λ_n as the lower limit of J under the conditions, it may be seen without difficulty that the series y_n contains all normal functions whose corresponding parameter value λ_n is positive or zero.

The values λ_n increase without limit, for they are the zeros of Δ, which is an integral transcendental function of λ.* If these zeros had a limiting value $\bar{\lambda}$ then Δ would vanish for $\lambda = \bar{\lambda}$ together with all its derivatives.

If A changes sign in the interval (x_1, x_2) the above method, with the single alteration that the equation

$$\int_{x_1}^{x_2} PAy^2\,dy = 1$$

be replaced by

$$\int_{x_1}^{x_2} PAy^2\,dx = -1,$$

shows the existence of an infinite series of negative normal parameter values λ_i and corresponding normal functions y_i. The results may be summarized in the following

THEOREM. *There exists an infinite series of normal parameter values λ_n and corresponding normal functions y_n for the differential equation*

$$(1') \qquad y'' + p(x)y' + [\lambda A(x) - B(x)]\,y = 0$$

under the boundary conditions

$$(2_0) \qquad \begin{cases} a_1 y(x_1) + a_2 y(x_2) + a_3 y'(x_1) + a_4 y'(x_2) = 0, \\ b_1 y(x_1) + b_2 y(x_2) + b_3 y'(x_1) + b_4 y'(x_2) = 0, \end{cases}$$

where, in writing $a_i b_k - a_k b_i = d_{ik}$,

$$e^{\int_{x_1}^{x_2} p\,dx}\, d_{13} = d_{24},$$

and where all the determinants d_{12}, d_{14}, d_{23}, d_{43} which are not zero have the same sign.

If the function $A(x)$ is not negative in the interval (x_1, x_2) the values λ_i are not negative, and increase without limit with i. If $A(x)$ changes sign in the interval the values λ_i include an infinite series of positive quantities increasing without limit and an infinite series of negative quantities decreasing without limit.

The function y_n minimizes the expression

$$J(y) = \int_{x_1}^{x_2} (y'^2 + By^2)\,P dx - [yy'P]_{x_1}^{x_2}$$

under the conditions

$$L_a(y) = 0, \qquad L_b(y) = 0,$$

* See PICARD, *Traité d'analyse*, t. 3 (1896), p. 92.

$$\int_{x_1}^{x_2} PAy^2\,dx = \pm 1,$$

$$\int_{x_1}^{x_2} PAy_i y\,dx = 0 \qquad [i = 0, \pm 1, \pm 2, \cdots, \pm(n-1)],$$

and gives to J the value $\pm\lambda_n$, the upper signs being taken if λ_n is positive, the lower signs if λ_n is negative.

§ 4. *An oscillation theorem for the periodic solutions.*

If the coefficients of (1′) are periodic functions with the period $x_2 - x_1$ then any solution of (1′) under the boundary conditions

$$y(x_1) - y(x_2) = 0,$$

$$y'(x_1) - y'(x_2) = 0,$$

is likewise periodic with the period $x_2 - x_1$. In this case, since $d_{12} = 0$, $d_{14} = -1$, $d_{23} = -1$, $d_{43} = 0$, conditions (16) are satisfied. Since $d_{13} = 1$, $d_{24} = 1$ we must have

$$\pi = e^{\int_{x_1}^{x_2} p\,dx} = 1,$$

or

$$\int_{x_1}^{x_2} p\,dx = 0,$$

in order to satisfy condition (3). It will be assumed that this condition is satisfied. Then there exists an infinite series of periodic solutions y_n of (1′) when the coefficients are periodic. We shall investigate the number of times y_n vanishes in the interval (x_1, x_2) under the assumption that

$$A(x) \geqq 0 \qquad (x_1 \leqq x \leqq x_2).$$

Let μ_n and u_n denote the normal parameter values and functions for the boundary condition

$$y(x_1) = 0, \qquad y(x_2) = 0.$$

Then, under the assumption $A \geqq 0$, according to STURM's theorems* the function u_n vanishes n times *within* the interval (x_1, x_1). We shall determine the number of times the periodic function y_n vanishes in (x_1, x_2) by means of this fact.

It may be proved from the definition of the normal parameter values as minimal values that

$$\lambda_n \leqq \mu_n \leqq \lambda_{n+1}.$$

*See BÔCHER, *Encyklopädie der mathematischen Wissenschaften*, II A 7a.

To show that $\mu_n \leqq \lambda_{n+1}$ it suffices to form a function w which satisfies the conditions

$$w(x_1) = 0, \qquad w(x_2) = 0,$$

$$\int_{x_1}^{x_2} PAw^2 dx = 1, \qquad \int_{x_1}^{x_2} PAu_i w dx = 0 \quad (i = 0, 1, 2, \cdots, n-1),$$

and gives to

$$\int_{x_1}^{x_2} (w'^2 + Bw^2) P dy$$

a value less than or equal to λ_{n+1}. Consider the function

$$w = c_0 y_0 + c_1 y_1 + \cdots + c_{n+1} y_{n+1},$$

formed from the periodic solutions y_i. Since $w(x_1) = w(x_2)$ the above boundary condition will be satisfied if $w(x_1) = 0$. Now the $n+2$ constants c_i may always be so determined that they are not all zero, and that the $n+1$ equations

$$w(x_1) = \Sigma c_k y_k(x_1) = 0,$$

$$\int_{x_1}^{x_2} PAu_i w dx = \Sigma c_k \int_{x_1}^{x_2} PAu_i y_k dx = 0 \quad (i = 0, 1, \cdots, n-1)$$

are satisfied, and furthermore so that

$$\int_{x_1}^{x_2} PAw^2 dx = \Sigma c_k^2 = 1.$$

Then w satisfies all the conditions of the minimum problem used in defining μ_n.

Multiplying the equation

$$(Py_i')' + (\lambda_i A - B) P y_i = 0$$

by y_k, integrating from x_1 to x_2 and remembering the conditions which y_n satisfies, we have

$$-\int_{x_1}^{x_2} Py_i' y_k' dx - \int_{x_1}^{x_2} PBy_i y_k dx = 0.$$

Hence

$$\int_{x_1}^{x_2} (w'^2 + Bw^2) P dx = \Sigma c_k^2 \int_{x_1}^{x_2} (y_k'^2 + By_k^2) P dx = \Sigma c_k^2 \lambda_k \leqq \lambda_{n+1} \Sigma c_k^2 = \lambda_{n+1}.$$

Therefore

$$\mu_n \leqq \lambda_{n+1}.$$

To prove the inequality

$$\lambda_n \leqq \mu_n,$$

it is to be noticed that λ_n is the lower limit of the values of

$$J(y) = \int_{x_1}^{x_2} (y'^2 + By^2)\, P dx$$

under the conditions

$$\int_{x_1}^{x_2} PAy^2 dx = 1, \qquad \int_{x_1}^{x_2} PAy_i y dx = 0 \quad (i = 0, 1, 2, \cdots, n-1),$$

$$y(x_1) = y(x_2), \qquad y'(x_1) = y'(x_2),$$

and since $J(y)$ contains y' only in the integral this lower limit will be unchanged when the condition $y'(x_1) = y'(x_2)$ is omitted.

The function

$$w = c_0 u_0 + c_1 u_1 + \cdots + c_n u_n$$

satisfies the equation $w(x_1) = w(x_2)$ since it vanishes at x_1 and x_2. Then the $n + 1$ constants c_i can always be determined so that they are not all zero, and that the n equations hold:

$$\Sigma c_k \int_{x_1}^{x_2} (PAy_i u_k dx = 0 \qquad (i = 0, 1, 2, \cdots, n-1).$$

The common factor of the constants c_i may then be determined so that

$$\int_{x_1}^{x_2} PAw^2 dx = \Sigma c_k^2 = 1,$$

and the desired inequality follows as in the preceding case.

Suppose y and $\bar{y}$ are functions satisfying the equations

$$(Py')' + (\lambda A - B)Py = 0,$$

$$(P\bar{y}')' + (\bar{\lambda} A - B)P\bar{y} = 0.$$

Then, since $P(x_1) = 1$, $P(x_2) = 1$,

$$[\bar{y}y' - y\bar{y}']_a^b = (\bar{\lambda} - \lambda) \int_a^b AP y \bar{y} dx.$$

Suppose a and b are consecutive zeros of $\bar{y}$, and that $\bar{y}$ is positive between a and b. Then $\bar{y}'(a) > 0$, $\bar{y}'(b) < 0$, for $\bar{y}'$ cannot vanish with $\bar{y}$. Suppose further that $\lambda > \bar{\lambda}$ and write $(+)$ for a positive function or constant. Then, since $A > 0$,

$$(+)y(a) + (+)y(b) = -(+) \int_a^b (+) y dx.$$

Hence y must vanish between a and b. That is, between two zeros of a solu-

tion of (1′) for a value $\bar{\lambda}$ there lies at least one zero of a solution of (1′) for $\lambda > \bar{\lambda}$.*

Suppose now that $\lambda_i = \mu_n$. If y_i and u_n are linearly independent, then y_i does not vanish at x_1 or at x_2 and since by the well known STURM's theorem the zeros of two linearly independent solutions of the same differential equation of type (1′) separate each other y_i vanishes $n+1$ times within the interval (x_1, x_2). If y_i and u_n are linearly dependent y_i vanishes with u_n, that is, $n+1$ times in the interval $(x_1 \leqq x < x_2)$. But y_i is periodic and therefore $\lambda_i = \mu_n$ only when n is an odd number. Therefore we have from the inequalities $\lambda_n \leqq \mu_n \leqq \lambda_{n+1}$,

$$\mu_{2m-2} < \lambda_{2m-1} \leqq \mu_{2m-1} \leqq \lambda_{2m} < \mu_{2m}.$$

If $\lambda_{2m-1} = \mu_{2m-1}$ then y_{2m-1} vanishes $2m$ times in the interval $(x_1 \leqq x < x_2)$, as was seen above. If $\lambda_{2m-1} < \mu_{2m-1}$ then, since between each pair of zeros of y_{2m-1} one zero of u_{2m-1} must lie, y_{2m-1} vanishes at most $2m$ times in $(x_1 \leqq x < x_2)$. But since $\mu_{2m-2} < \lambda_{2m-1}$, y_{2m-1} vanishes at least $2m-1$ times in $(x_1 \leqq x < x_2)$, and since y_{2m-1} is periodic it follows that y_{2m-1} vanishes in all cases $2m$ times for $x_1 \leqq x < x_2$.

If $\lambda_{2m} = \mu_{2m-1}$ then y_{2m} vanishes $2m$ times in $(x_1 \leqq x < x_2)$. If $\mu_{2m-1} < \lambda_{2m}$ then y_{2m} has at least $2m$ zeros in $(x_1 \leqq x < x_2)$. But since $\lambda_{2m} < \mu_{2m}$, y_{2m} has at most $2m+1$ zeros in this interval, and since y_{2m} is periodic it follows that it vanishes in all cases exactly $2m$ times for $x_1 \leqq x < x_2$. These results may be summarized in the following

THEOREM. *There exists an infinite series of normal parameter values λ_n and corresponding periodic functions y_n with the period $x_2 - x_1$, which satisfy the differential equations*

$$y_n'' + py_n' + (\lambda_n A - B)y_n = 0,$$

where

$$\int_{x_1}^{x_2} p\,dx = 0, \qquad B \not\equiv 0,$$

and where p, A, B are periodic with the period $x_2 - x_1$. If $A(x) \not\equiv 0$ in the interval (x_1, x_2) the functions y_{2m} and y_{2m-1} vanish exactly $2m$ times for $x_1 \leqq x < x_2$.

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* This is a well known result of STURM.

THE RESOLUTION OF ANY COLLINEATION INTO PERSPECTIVE REFLECTIONS*

BY

M. W. HASKELL

The general formulæ of a collineation being

$$\rho y_i = \sum_{k=1}^{k=n} a_{ik} x_k,$$

I shall consider only collineations which have n distinct fixed points and can therefore be reduced to the *normal form*

$$\rho y_i = m_i x_i.$$

These multipliers m_i are the roots of the *characteristic equation* of the collineation

$$\begin{vmatrix} a_{11}-\rho & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22}-\rho & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33}-\rho & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn}-\rho \end{vmatrix} = 0.$$

Just as any substitution can be resolved into a product of simple transpositions, so it will be shown that any collineation can be resolved into a product of perspective reflections, i. e., involutory collineations, and the minimum number of such perspective reflections will be determined. For the sake of clearness, the cases of the line, plane and ordinary space will be treated separately; the result for space of n dimensions is then deduced easily.

By *definition* a perspective reflection is one which leaves invariant a point $\kappa(\kappa_1 : \kappa_2 : \cdots : \kappa_{n+1})$ and every point in a flat space of $n-1$ dimensions, not containing κ and denoted by $a(a_1 : a_2 : \cdots : a_{n+1})$ or by the equation

* Presented to the Society at the Congress of Arts and Science at St. Louis, September 22, 1904. Received for publication January 11, 1905.

$a_x \equiv a_1x_1 + a_2x_2 + \cdots + a_{n+1}x_{n+1} = 0$, and which converts any point into its harmonic conjugate with respect to the point κ and the space a.

Hence the general formulæ for such a collineation may be written

$$\rho y_i = a_\kappa x_i - 2\kappa_i a_x,$$

the determinant of which is equal to $-a_\kappa^{n+1}$. By such a collineation every quadric space of $n-1$ dimensions with respect to which κ and a are pole and polar is obviously converted into itself. A perspective reflection will ordinarily be denoted by T, or by $T(a, \kappa)$ if it is desirable to put the elements a and κ in evidence.

Linear transformations of a single variable.

We can hardly use the word collineation in this case, but the general formulæ can be used, and will be used for the sake of uniformity.

THEOREM. *The general linear transformation of a single variable can be resolved into the product of two perspective reflections of period two, and in* ∞^1 *ways.*

If the general linear transformation S be reduced to its normal form

$$S: \qquad \rho y_i = m_i x_i$$

and be multiplied by the transformation $T(a, \kappa)$ of period two

$$T: \qquad \rho y_i = a_\kappa x_i - 2\kappa_i a_x,$$

the product $ST = U$ will be

$$U: \qquad \rho y_1 = (-a_1\kappa_1 + a_2\kappa_2)m_1x_1 - 2a_2\kappa_1 m_2x_2,$$
$$\rho y_2 = -2a_1\kappa_2 m_1x_1 + (a_1\kappa_1 - a_2\kappa_2)m_2x_2.$$

This transformation U will itself be of period two if the roots of its characteristic equation

$$\rho^2 - \rho(a_1\kappa_1 - a_2\kappa_2)(m_2 - m_1) - a_\kappa^2 m_1 m_2 = 0$$

are equal and opposite, that is to say, if $\kappa_1 : \kappa_2 = a_2 : a_1$. Then T reduces to

$$y_1 : y_2 = a_2^2 x_2 : a_1^2 x_1$$

and U becomes

$$y_1 : y_2 = a_2^2 m_2 x_2 : a_1^2 m_1 x_1.$$

It is evident that U is of period two, and since $ST = U$, it follows that $S = UT$, which was to be proved. Moreover since $a_1 : a_2$ is arbitrary, excluding of course the two cases $a_1 : a_2 = 1 : 0$ and $a_1 : a_2 = 0 : 1$, the resolution can be effected in ∞^1 ways.

Collineations in the plane.

The product of two perspective reflections $T_1(a, \kappa)$ and $T_2(b, \lambda)$ will leave invariant the point μ where a and b meet and the two points on the line $(\kappa\lambda)$ which are harmonically conjugate with respect to both (a, κ) and (b, λ). If we take the triangle $(\kappa\lambda\mu)$ as coördinate triangle, the various elements involved will be

$$\kappa(1:0:0), \qquad \lambda(0:1:0),$$
$$a(a_1:a_2:0), \qquad b(b_1:b_2:0),$$

and the formulæ of the collineations T_1 and T_2 become

$$T_1: \quad \rho y_1 = -a_1x_1 - 2a_2x_2, \qquad T_2: \quad \rho y_1 = b_2x_1,$$
$$\rho y_2 = a_1x_2, \qquad\qquad\qquad\quad \rho y_2 = -2b_1x_1 - b_2x_2,$$
$$\rho y_3 = a_1x_3, \qquad\qquad\qquad\quad \rho y_3 = b_2x_3.$$

From this we deduce

$$T_1T_2: \quad \rho y_1 = -a_1b_2x_1 - 2a_2b_2x_2,$$
$$\rho y_2 = 2a_1b_1x_1 + (4a_2b_1 - a_1b_2)x_2,$$
$$\rho y_2 = a_1b_2x_3,$$

of which the characteristic equation is

$$(a_1b_2 - \rho)[\rho^2 - \rho(4a_2b_1 - a_1b_2) + (a_1b_2)^2] = 0.$$

In this equation the product of two of the roots is equal to the square of the third, $\rho_1\rho_2 = \rho_3^2$, and the collineation T_1T_2 is therefore reducible to the normal form

$$S: \quad \rho y_1 = m_1x_1, \qquad \rho y_2 = m_2x_2, \qquad \rho y_3 = x_3,$$

where $m_1m_2=1$. This collineation converts into itself every conic $x_1x_2+kx_3^2=0$, the point $(0:0:1)$ being the polar of the line $x_3 = 0$. Conversely:

Every plane collineation which leaves a conic invariant can be resolved into the product of two perspective reflections, and this may be effected in ∞^1 ways.

For it can be shown that every such collineation S can be reduced to the normal form S just given. With S we must compound a reflection $T_2(a, \kappa)$, where κ lies on $x_3 = 0$ and a passes through $(0:0:1)$, that is,

$$\kappa = \kappa(\kappa_1:\kappa_2:0), \qquad a = a(a_1:a_2:0), \qquad a_\kappa = a_1\kappa_1 + a_2\kappa_2.$$

The formulæ of T_2 are therefore

$$T_2: \quad \rho y_1 = (-a_1\kappa_1 + a_2\kappa_2)x_1 - 2a_2\kappa_1x_2,$$
$$\rho y_2 = -2a_1\kappa_2x_1 + (a_1\kappa_1 - a_2\kappa_2)x_2,$$
$$\rho y_3 = (a_1\kappa_1 + a_2\kappa_2)x_3.$$

and the characteristic equation of ST_2 is

$$(a_\kappa - \rho)[\rho^2 - \rho(a_1\kappa_1 - a_2\kappa_2)(m_2 - m_1) - a_\kappa^2] = 0.$$

Hence $ST_2 = T_1$ will itself be a perspective reflection if $a_1\kappa_1 - a_2\kappa_2 = 0$, that is, if $a_1 : a_2 = 1/\kappa_1 : 1/\kappa_2$, where $\kappa_1 : \kappa_2$ is arbitrary, only the points $1:0:0$ and $0:1:0$ being excluded. From $ST_2 = T_1$ and $T_2^2 = 1$, follows $S = T_1 T_2$, and it is clear that this resolution may be effected in ∞^1 ways.

It will now be shown that corresponding to any plane collineation U, there are ∞^3 perspective reflections T_3 such that the product $UT_3 = S$ will be of the type just considered, leaving a conic unchanged. If U be written in the normal form

$$U: \qquad \rho y_i = m_i x_i$$

and T_3 in the general form

$$T_3: \qquad \rho y_i = a_\kappa x_i - 2\kappa_i a_x,$$

the formulæ for UT_3 become

$$\begin{aligned}
\rho y_1 &= (a_\kappa - 2a_1\kappa_1)m_1 x_1 - 2a_2\kappa_1 m_2 x_2 - 2a_3\kappa_1 m_3 x_3,\\
\rho y_2 &= -2a_1\kappa_2 m_1 x_1 + (a_\kappa - 2a_2\kappa_2)m_2 x_2 - 2a_3\kappa_2 m_3 x_3,\\
\rho y_3 &= -2a_1\kappa_3 m_1 x_1 - 2a_2\kappa_3 m_2 x_2 + (a_\kappa - 2a_3\kappa_3)m_3 x_3.
\end{aligned}$$

The characteristic equation of this collineation is

$$\begin{aligned}
-\rho^3 + \rho^2[(a_\kappa - 2a_1\kappa_1)m_1 + (a_\kappa - 2a_2\kappa_2)m_2 + (a_\kappa - 2a_2\kappa_3)m_3]\\
- \rho a_\kappa[m_2 m_3(a_\kappa - 2a_2\kappa_2 - 2a_3\kappa_3) + m_3 m_1(a_\kappa - 2a_3\kappa_3 - 2a_1\kappa_1)\\
+ m_1 m_2(a_\kappa - 2a_1\kappa_1 - 2a_2\kappa_2)] - a_\kappa^3 m_1 m_2 m_3 = 0.
\end{aligned}$$

If now UT_3 is to leave a conic unchanged, the roots of this characteristic equation must satisfy the relation $\rho_1\rho_2 = \rho_3^2$, whence

$$(\rho_1\rho_2 + \rho_2\rho_3 + \rho_3\rho_1)^3 = \rho_1\rho_2\rho_3(\rho_1 + \rho_2 + \rho_3)^3,$$

or the coefficients of the equation satisfy the relation

$$[\Sigma m_2 m_3(a_\kappa - 2a_2\kappa_2 - 2a_3\kappa_3)]^3 + m_1 m_2 m_3[\Sigma m_1(a_\kappa - 2a_1\kappa_1)]^3 = 0.$$

Here κ may be chosen arbitrarily, excluding only the fixed points of U, and then this condition becomes the equation of three points, through one of which a must pass. Then T_3 may be found in $\infty^2 \cdot \infty^1 = \infty^3$ ways so that $UT_3 = S$ shall be of the required type.

Combining these two results, we have the following

THEOREM. *Any plane collineation U can be resolved in ∞^4 ways into the product of three perspective reflections $T_1 T_2 T_3$.*

A case of especial interest and importance in that in which

$$\sum m_1(a_\kappa - 2a_1\kappa_1) = 0, \qquad \sum m_2 m_3(a_\kappa - 2a_2\kappa_2 - 2a_3\kappa_3) = 0,$$

for in this case $S^3 = 1$. The solution of these equations gives

$$a_1\kappa_1 : a_2\kappa_2 : a_3\kappa_3$$
$$= (m_2 - m_3)(m_2 m_3 + m_1^2) : (m_3 - m_1)(m_3 m_1 + m_2^2) : (m_1 - m_2)(m_1 m_2 + m_3^2).$$

A number of well-known groups are generated by collineations S and T, where

$$S^n = 1, \qquad T^2 = 1, \qquad (ST)^3 = 1,$$

and the analysis just given readily furnishes the complete formulæ.

Collineations in space of three dimensions.

The product of two reflections in space $T_1(a, \kappa)$ and $T_2(b, \lambda)$ will leave invariant every point on the line of intersection of the planes a and b, together with two points on the line $(\kappa\lambda)$. It can be shown just as above that the normal form of the product $T_1 T_2$ will be

$$y_1 : y_2 : y_3 : y_4 = m_1 x_1 : m_2 x_2 : x_3 : x_4,$$

where $m_1 m_2 = 1$.

The product of three perspective reflections $T_1(a, \kappa)$, $T_2(b, \lambda)$ and $T_3(c, \mu)$ will leave invariant the point ν common to a, b and c, and also three points on the plane $(\kappa\lambda\mu)$. Taking $\kappa, \lambda, \mu, \nu$ as the vertices of the tetrahedron of reference, we have

$$\kappa = \kappa(1:0:0:0), \qquad \lambda = \lambda(0:1:0:0), \qquad \mu = \mu(0:0:1:0),$$
$$a = a(a_1 : a_2 : a_3 : 0), \qquad b = b(b_1 : b_2 : b_3 : 0), \qquad c = c(c_1 : c_2 : c_3 : 0),$$

and the formulæ of T_1 are simply

$$\begin{aligned} T_1: \qquad \rho y_1 &= -a_1 x_1 - 2a_2 x_2 - 2a_3 x_3, \\ \rho y_2 &= a_1 x_2, \\ \rho y_3 &= a_1 x_3, \\ \rho y_4 &= a_1 x_4, \end{aligned}$$

with similar formulæ for T_2 and T_3. Hence we have for $T_1 T_2 T_3$ the formulæ

$$\begin{aligned} \rho y_1 &= -a_1 b_2 c_3 x_1 - 2a_2 b_2 c_3 x_2 - 2a_3 b_2 c_3 x_3, \\ \rho y_2 &= 2a_1 b_1 c_3 x_1 + (4a_2 b_1 c_3 - a_1 b_2 c_3) x_2 + (4a_3 b_1 c_3 - 2a_1 b_3 c_3) x_3, \\ \rho y_3 &= (2a_1 b_2 c_1 - 4a_1 b_1 c_2) x_1 + (4a_2 b_2 c_1 - 8a_2 b_1 c_2 + 2a_1 b_2 c_2) x_2 \\ &\qquad + (4a_3 b_2 c_1 - 8a_3 b_1 c_2 + 4a_1 b_3 c_2 - a_1 b_2 c_3) x_3, \\ \rho y_4 &= a_1 b_2 c_3 x_4; \end{aligned}$$

and the characteristic equation is

$$(a_1b_2c_3-\rho)[-\rho^3+\rho^2(-8a_3b_1c_2+4a_1b_3c_2+4a_3b_2c_1+4a_2b_1c_3-3a_1b_2c_3)$$
$$-\rho(a_1b_2c_3)(8a_2b_3c_1-4a_1b_3c_2-4a_3b_2c_1-4a_2b_1c_3+3a_1b_3c_3)-(a_1b_2c_3)^3]=0.$$

This equation is characterized by the property that $\rho_1\rho_2\rho_3+\rho_4^3=0$, and the normal form of the collineation will be $y_1:y_2:y_3:y_4=m_1x_1:m_2x_2:m_3x_3:-x_4$, where $m_1m_2m_3=1$.

I propose now to show that any collineation S_4 in space of three dimensions can be reduced to a product S_3T_4 where the multipliers of S_3 are connected by the relation $m_1m_2m_3+m_4^3=0$ and T_4 is a reflection; that a collineation of type S_3 can be reduced to a product S_2T_3 where the multipliers of S_2 satisfy the relations $m_1m_2=m_3^2$, $m_3=m_4$, and T_3 is a reflection; finally, that a collineation of type S_2 can be reduced to a product of two reflections T_1T_2, and hence that $S_4=T_1T_2T_3T_4$.

If S_4 be reduced to the normal form

$$\rho y_i=m_ix_i$$

and T_4 be given in the general form

$$\rho y_i=a_\kappa x_i-2\kappa_i a_x,$$

then $S_4T_4=S_3$ will be of the form

$$\rho y_i=a_\kappa m_ix_i-2\kappa_i a_{mx},$$

and the characteristic equation of S_3 will be

$$\begin{vmatrix}(a_\kappa-2a_1\kappa_1)m_1-\rho & -2a_2\kappa_1m_2 & -2a_3\kappa_1m_3 & -2a_4\kappa_1m_4\\ -2a_1\kappa_2m_1 & (a_\kappa-2a_2\kappa_2)m_2-\rho & -2a_3\kappa_2m_3 & -2a_4\kappa_2m_4\\ -2a_1\kappa_3m_1 & -2a_2\kappa_3m_2 & (a_\kappa-2a_3\kappa_3)m_3-\rho & -2a_4\kappa_3m_4\\ -2a_1\kappa_4m_1 & -2a_2\kappa_4m_2 & -2a_3\kappa_4m_3 & (a_\kappa-2a_4\kappa_4)m_4-\rho\end{vmatrix}=0.$$

This reduces to

$$\rho^4-\rho^3\sum(a_\kappa-2a_1\kappa_1)m_1+\rho^2a_\kappa\sum m_1m_2(a_\kappa-2a_1\kappa_1-2a_2\kappa_2)$$
$$-\rho a_\kappa^2\sum m_1m_2m_3(a_\kappa-2a_1\kappa_1-2a_2\kappa_2-2a_3\kappa_3)-a_\kappa^4m_1m_2m_3m_4=0,$$

and S_3 will be of the required character if this equation be satisfied by any fourth root of $a_\kappa^4m_1m_2m_3m_4$; T_4 can therefore be chosen in ∞^5 different ways.

In particular, if the equations

$$\sum m_1(a_\kappa-2a_1\kappa_1)=0,$$
$$\sum m_1m_2(a_\kappa-2a_1\kappa_1-2a_2\kappa_2)=0,$$
$$\sum m_1m_2m_3(a_\kappa-2a_1\kappa_1-2a_2\kappa_2-2a_3\kappa_3)=0$$

be simultaneously satisfied, $S_3^1 = 1$; and in this case T_4 can be chosen in ∞^3 ways.

Reducing S_3 to its normal form

$$y_1 : y_2 : y_3 : y_4 = m_1 x_1 : m_2 x_2 : m_3 x_3 : -x_4,$$

where $m_1 m_2 m_3 = 1$, we must choose the elements of $T_3(b, \lambda)$, so that λ lies on $x_4 = 0$ and b passes through $0:0:0:1$, that is to say,

$$\lambda = \lambda(\lambda_1 : \lambda_2 : \lambda_3 : 0) \qquad \text{and} \qquad b = b(b_1 : b_2 : b_3 : 0),$$

so that $b_\lambda \equiv b_1\lambda_1 + b_2\lambda_2 + b_3\lambda_3$. We can then write down the characteristic equation of $S_3 T_3$ immediately as follows:

$$\begin{vmatrix} (b_\lambda - 2b_1\lambda_1)m_1 - \rho & -2b_2\lambda_1 m_2 & -2b_3\lambda_1 m_3 & 0 \\ -2b_1\lambda_2 m_1 & (b_\lambda - 2b_2\lambda_2)m_2 - \rho & -2b_3\lambda_2 m_3 & 0 \\ -2b_1\lambda_3 m_1 & -2b_2\lambda_3 m_2 & (b_\lambda - 2b_2\lambda_2)m_3 - \rho & 0 \\ 0 & 0 & 0 & -b_\lambda - \rho \end{vmatrix} = 0.$$

This is equivalent to

$$-(b_\lambda + \rho)[-\rho^3 + \rho^2 \Sigma m_1(b_\lambda - 2b_1\lambda_1) - \rho b_\lambda \Sigma m_1 m_2(b_\lambda - 2b_1\lambda_1 - 2b_2\lambda_2) - b_\lambda^3] = 0.$$

One root of this equation is obviously $-b_\lambda$; and, that the product $S_3 T_3 = S_2$ be of the required form, it is necessary and sufficient that a second root should also be equal to $-b_\lambda$; hence

$$\Sigma m_1(b_\lambda - 2b_1\lambda_1) + \Sigma m_1 m_2(b_\lambda - 2b_1\lambda_1 - 2b_2\lambda_2) = 0.$$

This reduction can evidently be performed in ∞^3 ways, and S_2 can be reduced to the normal form

$$y_1 : y_2 : y_3 : y_4 = m_1 x_1 : m_2 x_2 : x_3 : x_4,$$

where $m_1 m_2 = 1$.

In particular, if

$$\Sigma m_1(b_\lambda - 2b_1\lambda_1) = 0,$$

$$\Sigma m_1 m_2(b_\lambda - 2b_1\lambda_1 - 2b_2\lambda_2) = 0,$$

S_2 will be of period three; and this reduction can be effected in ∞^2 ways.

The elements of $T_2(c, \mu)$ must now be so chosen that μ lies on the line $x_3 = 0$, $x_4 = 0$, and c passes through the line $x_1 = 0$, $x_2 = 0$. Then

$$\mu = \mu(\mu_1 : \mu_2 : 0 : 0) \qquad \text{and} \qquad c = c(c_1 : c_2 : 0 : 0),$$

so that $c_\mu = c_1\mu_1 + c_2\mu_2$. The characteristic equation of $S_2 T_2 = T_1$ is then

$$\begin{vmatrix} (c_\mu - 2c_1\mu_1)m_1 - \rho & -2c_2\mu_1 m_2 & 0 & 0 \\ -2c_1\mu_2 m_1 & (c_\mu - 2c_2\mu_2)m_2 - \rho & 0 & 0 \\ 0 & 0 & c_\mu - \rho & 0 \\ 0 & 0 & 0 & c_\mu - \rho \end{vmatrix} = 0$$

and this is equivalent to

$$(c_\mu - \rho)^2 [\rho^2 - \rho\Sigma m_1(c_\mu - 2c_1\mu_1) - c_\mu^2] = 0.$$

T_1 will be a reflection if the coefficient of ρ is equal to zero. This gives

$$c_1\mu_1 - c_2\mu_2 = 0,$$

and the reduction can be performed in ∞^1 ways.

We have then the final result:

THEOREM. *Any collineation in space may be reduced to a product of four perspective reflections in ∞^9 ways. In particular, the reduction $S_4 = T_1 T_2 T_3 T_4$, subject to the relations $(T_1 T_2)^3 = 1$, $(T_1 T_2 T_3)^4 = 1$ can be effected in ∞^6 ways.*

IV. *Collineations in space of n dimensions.*

Passing now to the general case of collineations in space of n dimensions we observe first that the product of $k+1$ reflections ($k \leqq n$),

$$T_1(a, \kappa), \qquad T_2(b, \lambda), \qquad \cdots \qquad T_{k+1}(s, \sigma)$$

leaves invariant all the points common to the spaces $a, b, \cdots, s$, together with $k+1$ points lying in the flat space of k dimensions determined by the points $\kappa, \lambda, \cdots, \sigma$. The normal form of such a product is evidently

$$\rho y_i = m_i x_i \qquad (i = 1, 2, 3, \cdots, k+1),$$

$$\rho y_i = x_i \qquad (i = k+2, k+3, \cdots, n+1),$$

where $m_1 m_2 \cdots m_{k+1} = (-1)^{k+1}$.

Vice versa, any collineation S_{k+1} which can be reduced to the above normal form can be resolved in ∞^{k^2} ways into the product of $k+1$ perspective reflections, as I shall show immediately. We must evidently compound S_{k+1} first with a perspective reflection $T_{k+1}(s, \sigma)$ for which

$$s = s(s_1 : s_2 : \cdots s_{k+1} : 0 : 0 : \cdots 0),$$

$$\sigma = \sigma(\sigma_1 : \sigma_2 : \cdots \sigma_{k+1} : 0 : 0 : \cdots 0),$$

and hence

$$s_\sigma = s_1\sigma_1 + s_2\sigma_2 + \cdots s_{k+1}\sigma_{k+1}.$$

Then the characteristic equation of $S_{k+1}T_{k+1}$ will be

$$(s_\sigma-\rho)^{n-k}\left[(-\rho)^{k+1}+(-\rho)^k\sum m_1(s_\sigma-2s_1\sigma_1)\right.$$
$$+(-\rho)^{k-1}s_\sigma\sum m_1m_2(s_\sigma-2s_1\sigma_1-2s_2\sigma_0)$$
$$\left.+\cdots-m_1m_2\cdots m_{k+1}s_\sigma^{k+1}\right]=0;$$

$(n-k)$ roots of this equation are obviously $=s_\sigma$; still another root will have this value if

$$\sum m_1(s_\sigma-2s_1\sigma_1)-\sum m_1m_2(s_\sigma-2s_1\sigma_1-2s_2\sigma_2)+\cdots$$
$$+(-1)^{k-1}\sum m_1m_2\cdots m_k(s_\sigma-2s_1\sigma_1\cdots 2s_k\sigma_k)=0,$$

where evidently σ may be chosen arbitrarily, that is, in ∞^k ways, and s can then be chosen in ∞^{k-1} ways. The normal form of $S_{k+1}T_{k+1}=S_k$ is then of the same form as that of S_{k+1}, k being substituted for $k+1$, and the reduction can be performed in ∞^{2k-1} ways.

In particular, the collineation S_k will be of period $k+1$ if all the terms in the above equation vanish simultaneously:

$$\sum m_1(s_\sigma-2s_1\sigma_1)=0,$$
$$\sum m_1m_2(s_\sigma-2s_1\sigma_1-2s_2\sigma_2)=0,$$
$$\text{etc.}$$

In this case, there are exactly enough equations to determine the ratios

$$s_1\sigma_1:s_2\sigma_2:\cdots:s_{k+1}\sigma_{k+1},$$

so that σ may be chosen arbitrarily and s is then determined. The reduction is then possible in ∞^k ways.

Continuing this process step by step, we arrive at the general

THEOREM. *A collineation of type S_{k+1} defined by the formulæ*

$$S_{k+1}: \qquad \rho y_i=m_ix_i \qquad (i=1,2,\cdots,k+1),$$
$$\rho y_i=x_i \qquad (i=k+2,\cdots,n+1),$$

where $m_1m_2\cdots m_{k+1}=(-1)^{k+1}$ can be resolved into a product of $k+1$ perspective reflections $T_1T_2T_3\cdots T_{k+1}$ in ∞^{k^2} ways. If these reflections be subject to the conditions

$$(T_1T_2)^3=1, \qquad (T_1T_2T_3)^4=1, \text{ etc.},$$

the reduction can be effected in $\infty^{\frac{1}{2}k(k+1)}$ ways. Moreover, it is clear that $k+1$ is the minimum number of reflections involved in the reduction.

The general collineation in space of n dimensions is included in the theorem; it is only necessary to let $k=n$.

LINEAR ALGEBRAS IN WHICH DIVISION IS ALWAYS UNIQUELY POSSIBLE*

BY

LEONARD EUGENE DICKSON

§ 1. *Introduction; summary of results.*

Let the elements of an algebra be $A = \sum_{i=1}^{m} a_i e_i$, where the coördinates a_i range over a given field F, while the units e_i are linearly independent with respect to F and have a multiplication table,

$$e_i e_j = \sum_{k=1}^{m} \gamma_{ijk} e_k \qquad (i, j = 1, \cdots, m\,;\ \gamma\text{'s in } F).$$

Given two elements A and B with the a_i not all zero, we can determine an unique element X_r such that $AX_r = B$ and an unique element X_l such that $X_l A = B$ if and only if each of the determinants

$$\Delta_r \equiv \left| \sum_{i=1}^{m} \gamma_{ijk} a_i \right|, \qquad \Delta_l \equiv \left| \sum_{i=1}^{m} \gamma_{jik} a_i \right| \qquad (j, k = 1, \cdots, m)$$

does not vanish. Hence the condition that right hand [left hand] division shall always be uniquely possible is that Δ_r [Δ_l] shall vanish only when every a_i vanishes. Now either of these conditions is satisfied when the other is, since either is equivalent to the condition that a product shall vanish only when one factor vanishes. We consider algebras in which these conditions are satisfied and in which there is a modulus, i. e., an element 1 such that $1A = A1 = A$ for every element A. We shall henceforth set $e_1 = 1$.

For $m = 2$, the algebra is the field $F(e_2)$. Indeed, $e_2^2 - e_2\gamma_{222} - \gamma_{221} = 0$ is irreducible in F since $\Delta_r = a_1^2 + a_1 a_2 \gamma_{222} - a_2^2 \gamma_{221}$.

In § 2 I consider the general transformation of algebras with three units, exhibiting families of algebras invariant under every linear transformation and determining the algebras which admit more than one transformation into itself (and hence exactly three transformations). From each standpoint I am led to the same remarkable set of families of algebras, each set characterized by a parameter μ. For $\mu = 1$, the family consists of all fields of rank three with respect to F. For $\mu = 0$, the commutative algebras have the property that division is always possible.

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In § 3 there is indicated a method of deducing an algebra of mk units from one of m units. In §§ 4, 5, there is indicated a method of constructing a remarkable algebra in m units, where m is any even integer greater than two, such that division is always possible. An important point in the theory is established in § 6 by the exhibition of two non-equivalent, non-field, commutative algebras in six units with coördinates in the same general field.

Throughout the paper I exclude fields with modulus 2.

§ 2. *Algebras with three units*, $1, i, j$.

Suppose first that, for every element e, e^2 is a linear function of e. Then $i^2 = ai + A$, $j^2 = bi + B$. Set $I = i - a/2$, $J = j - b/2$. Hence

$$I^2 = \alpha, \qquad J^2 = \beta, \qquad IJ = r + sI + tJ, \qquad JI = R + SI + TJ.$$

Applying the assumption to $e = I + lJ$ for $l = 1$ and -1 in turn, we find that $IJ + JI$ must be a constant. Hence $S = -s$, $T = -t$. We may assume that s and t are not both zero, since otherwise $(RI - \alpha J)I = 0$. We show that division by $E \equiv sI + tJ$ is not possible in general. If we express $(x + yI + zJ)E$ in the form $K + LI + MJ$, we find that $tL \equiv sM$. Hence arbitrary values cannot be given to both L and M.

Hence there exists an element e for which e^2 is linearly independent of e. We may therefore set $i = e, j = e^2$. Then

$$\text{(I)} \qquad i^2 = j, \quad ij = b + \beta i + Bj, \quad ji = a + \alpha i + Aj, \quad j^2 = d + \delta i + Dj.$$

If $a = 0$ or $b = 0$ there exists no solution of $Xi = 1$ or $iX = 1$, respectively. Hence $a \neq 0, b \neq 0$. Suppose that $x^3 - b - \beta x - Bx^2 = 0$ has a root $x = -w$ in the field, and set $I = i + w$, $J = j + 2iw + w^2$. Then

$$I^2 = J, \qquad IJ = b - \beta w + Bw^2 + w^3 + (\beta - 2Bw - 3w^2)I + (B + 3w)J.$$

The constant term in IJ thus vanishes and the algebra is excluded. Similarly, the constant term in JI is $a - \alpha w + Aw^2 + w^3$ and hence would vanish if $-w$ is a root of $x^3 - a - \alpha x - Ax^2 = 0$. *Hence any triple algebra in a field F not having modulus 2 may be given the form* (I) *with $x^3 - b - \beta x - Bx^2$ and $x^3 - a - \alpha x - Ax^2$ irreducible in F.*

To algebra (I) we apply the transformation of units,

$$\text{(1)} \quad I = r + si + tj, \quad J = I^2 = \omega + \kappa i + \rho j \quad \left(\begin{matrix} \omega = r^2 + st(a+b) + t^2 d, \\ \kappa = 2rs + st(\alpha + \beta) + t^2 \delta, \\ \rho = s^2 + 2rt + st(A + B) + t^2 D \end{matrix} \right).$$

This transformation is valid when r, s, t are any marks such that s and t are not both zero. In fact, we show that

$$C \equiv s\rho - t\kappa \equiv s^3 + s^2 t(A + B) + st^2(D - \alpha - \beta) - t^3 \delta$$

vanishes only when $s=t=0$. Replacing a_1, a_2, a_3 by r, s, t, respectively, we get

$$\Delta_r = \begin{vmatrix} r & ta & sb+td \\ s & r+t\alpha & s\beta+t\delta \\ t & s+tA & r+sB+tD \end{vmatrix}.$$

Denote the minors of the elements of the first row in Δ_r by $\Delta', \Delta'', \Delta'''$. The latter, and hence also Δ_r, can be made to vanish if there exists a set of solutions s and t, not both zero, of $C=0$. Indeed, we then have $t \neq 0$ and we can determine r uniquely to satisfy $\Delta'''=0$. But upon substituting this value of r in $t\Delta''=0$ and $t^2\Delta'=0$, we get $C=0$ and $(s+At)C=0$, respectively.

Under the transformation (1) algebra (I) becomes.*

$$I^2=J, \qquad IJ=b'+\beta' I+B'J, \qquad JI=a'+\alpha' I+A'J, \qquad J^2=d'+\delta' I+D'J,$$

$$B'=3r+sB+t(\alpha+D), \qquad A'=3r+sA+t(\beta+D),$$

$$\beta'=-3r^2-2rt(\alpha+D)-2rsB+s^2\beta+st(a+b+\delta+\beta A-\alpha B)+t^2(d+\delta A-\alpha D),$$

$$\alpha'=-3r^2-2rt(\beta+D)-2rsA+s^2\alpha+st(a+b+\delta+\alpha B-\beta A)+t^2(d+\delta B-\beta D),$$

$$b'=r^3+r^2t(\alpha+D)+r^2sB-rs^2\beta-rst(\alpha+b+\delta+\beta A-\alpha B)-rt^2(d+\delta A-aD)+s^3b+s^2t(d+bA-aB)+st^2(dA-aD+a\beta-b\alpha)+t^3(a\delta-\alpha d),$$

$$a'=r^3+r^2t(\beta+D)+r^2sA-rs^2\alpha-rst(a+b+\delta+\alpha B-\beta A)-rt^2(d+\delta B-\beta D)+s^3a+s^2t(d+aB-bA)+st^2(dB-bD+b\alpha-a\beta)+t^3(b\delta-d\beta),$$

$$D'=6r^2+2rs(A+B)+2rt(\alpha+\beta+2D)+s^2D+st[2a+2b-\delta+(A+B)(\alpha+\beta+D)]+t^2(D^2+2d+D\alpha+D\beta),$$

$$\delta'=-8r^3-4r^2s(A+B)-4r^2t(\alpha+\beta+2D)+2rs^2(\alpha+\beta-D)+rst[6\delta-2(A+B)(\alpha+\beta+D)]+rt^2[2\delta(A+B)-4D(\alpha+\beta)-2D^2]+s^3\delta+s^2t[2\delta(A+B)+(\alpha+\beta)(\alpha+\beta-D)]+st^2[\delta D+2\delta(\alpha+\beta)+\delta(A+B)^2-D(\alpha+\beta)(A+B)]+t^3[\delta^2+\delta D(A+B)-D^2(\alpha+\beta)],$$

$$d'=3r^4+2r^3s(A+B)+2r^3t(\alpha+\beta+2D)+r^2s^2(D-2\alpha-2\beta)+r^2t^2[D^2+3D(\alpha+\beta)-2d-2\delta(A+B)]+r^2st[(A+B)(\alpha+\beta+D)-2a-2b-5\delta]$$

* The simplest method of performing this computation is to identify the linear expressions in i, j for IJ and $b'+\beta' I+B'J$, etc. Then α', β' and D may be found by a single division by C; while a', b', d' follow without such a division. The computation was checked in various ways.

$$
\begin{aligned}
&- rs^2t\,[(\alpha+\beta)(\alpha+\beta-D)+2\delta(A+B)-4d]+rs^3(2a+2b-\delta)\\
&- rst^2[\delta D+2\delta(\alpha+\beta)+\delta(A+B)^2-2d(A+B)+2D(a+b)\\
&- D(\alpha+\beta)(A+B)]-rt^3[\delta^2+\delta D(A+B)-D^2(\alpha+\beta)\\
&+2d(\alpha+\beta)-2\delta(a+b)]-t^4[d^2+dD(\alpha+\beta)-\delta D(a+b)]\\
&+st^3[\delta(a+b)(A+B)-2d(a+b)-D^2(a+b)+\delta d+dD(A+B)\\
&- d(\alpha+\beta)(A+B)]+s^2t^2[dD+d(A+B)^2-(a+b)\gamma+2\delta(a+b)\\
&- D(a+b)(A+B)]+s^3t[2d(A+B)+(a+b)(\alpha+\beta-D)]+s^4d.
\end{aligned}
$$

The set of all algebras (I) in which division is uniquely possible is invariant under every transformation (1) with r and s not both zero. This property evidently holds for the set of commutative algebras. It holds for the larger set of all algebras (I) with $A=B$, $\alpha=\beta$, since then $A'=B'$, $\alpha'=\beta'$. Within the latter set of algebras there exist sub-sets such that each is invariant under the transformations (1) and such that d, δ and D are fixed rational integral functions of a, b, β, B. The totality of all fields (I_1) obviously forms such a sub-set. We proceed to determine all such sub-sets. Under the assumption that $A=B$, $\alpha=\beta$, we require that d', δ', D' shall be expressible as rational integral functions of a', b', β', B', with coefficients independent of r, s, t, a, b, β, B. Thus must $D'=\lambda\beta'+\mu B'^2$, whence

$$
9\mu-3\lambda-6=0,\quad (\beta+D)(6\mu-2\lambda-4)=0,\quad B(6\mu-2\lambda-4)=0,
$$

$$
D=\lambda\beta+\mu B^2,\quad (\lambda-2)(a+b)+(\lambda+1)\delta+2\mu B(\beta+D)-2B(2\beta+D)=0,
$$

$$
(\lambda-2)d+\lambda(\delta B-\beta D)+(\mu-1)(D+\beta)^2+\beta^2=0.
$$

In case F has modulus 3 we assume * here and also below that F is the $GF[3^n]$. Then if $\beta+D=0$ and $B=0$, $C\equiv s^3-t^3\delta$ would be reducible. Hence, in every case,

$$
(2)\quad \begin{cases} D=(3\mu-2)\beta+\mu B^2, \\ (a+b)(3\mu-4)+\delta(3\mu-1)+2\beta B\mu(3\mu-4)+2B^3\mu(\mu-1)=0, \\ d(3\mu-4)+\delta B(3\mu-2)+B^4\mu^2(\mu-1) \\ \qquad +B^2\beta\mu(3\mu-4)(2\mu-1)+\beta^2(\mu-1)(3\mu-1)(3\mu-4)=0. \end{cases}
$$

Now it may be verified that also the last two relations hold true for the transformed algebra. For each value of μ, *the set of all the algebras* (I) *in which* $A=B$, $\alpha=\beta$, *and relations* (2) *hold is invariant under every transformation* (1) *with* r *and* s *not both zero.* Within each set, the sub-sets of algebras

* Note that there exist infinite fields having modulus 3 such that there is an irreducible function of the form x^3-z, e. g., the totality of rational functions of a parameter z, with integral coefficients taken modulo 3.

with $a=b$ is invariant. The sub-sets given by $\mu=0$ and $\mu=1$ are respectively

(I_0) $\quad i^2=j,\ ij=ji=b+\beta i+Bj,\ j^2=4bB-\beta^2-8bi-2\beta j;$

(I_1) $\quad i^2=j,\ ij=ji=b+\beta i+Bj,\ j^2=bB+(b+\beta B)i+(\beta+B^2)j.$

Let F be any field (not having modulus 2) for which

$$f(x)\equiv x^3-b-\beta x-Bx^2$$

is irreducible. Then algebra (I_1) is the *field* $F(i)$. We proceed to prove that the non-field * algebra (I_0) has the property that division is always uniquely possible. For (I_0) and (I_1), Δ_r is, respectively,

$$\begin{vmatrix} r & tb & sb+t(4bB-\beta^2) \\ s & r+t\beta & s\beta+t(-8b) \\ t & s+tB & r+sB+t(-2\beta) \end{vmatrix}, \qquad \begin{vmatrix} r & tb & sb+tbB \\ s & r+t\beta & s\beta+t(b+\beta B) \\ t & s+tB & r+sB+t(\beta+B^2) \end{vmatrix}.$$

Now the latter is transformed into the former by the replacement of r, s, t by $r+\beta t, s+2Bt, -2t$, respectively, the transformation being valid since F does not have modulus 2. But for a field, Δ vanishes only when $r=t=s=0$. *Division is always uniquely possible in the non-field algebra* (I_0), *where* b, β, B *are any marks of any field* (*not having modulus* 2) *for which* $x^3-b-\beta x-Bx^2$ *is irreducible.* †

Another remarkable family of algebras invariant under every transformation (1) is the set of all algebras (I) for which

$$A=B, \qquad \alpha=\beta=-\tfrac{1}{3}B^2, \qquad a+b+\delta+\tfrac{2}{3}B(D+\beta)=0,$$
$$d+\delta B-\beta D+\tfrac{1}{3}(D+\beta)^2=0.$$

The last two equations may be given the form

$$\delta=-a-b-\tfrac{2}{3}B(D-\tfrac{1}{3}B^2), \qquad d=B(a+b)-\tfrac{1}{3}D^2+\tfrac{5}{9}DB^2-\tfrac{7}{27}B^4,$$

so that $A, \alpha, \beta, \delta, d$ are given functions of a, b, B, D. But in this algebra division is not uniquely possible for an *arbitrary* field F. Indeed, for $x=y+\frac{1}{3}B$,

$$f(x)\equiv x^3-b+\tfrac{1}{3}B^2x-Bx^2=y^3+\tfrac{1}{27}B^3-b.$$

But if F is the $GF[p^n]$, and p^n is of the form $3l+2$, every mark is a cube, so that $f(x)$ is reducible for every b and B.

*Suppose, indeed, that algebras (I_0) and (I_1) are identical. Then, if F does not have modulus 3, $\beta=-\frac{1}{3}B^2$, $b=\frac{1}{27}B^3$, so that $f(x)$ vanishes for $x=\frac{1}{3}B$. If F has modulus 3, then $\beta=B=0$, and x^3-b is reducible in the $GF[3^n]$.

† For the case $B=0$, this algebra was given in my memoir on finite algebras, Göttingen Nachrichten, 1905, pp. 358-393.

We next discuss the equivalence (under a linear transformation of units) of algebras (I). Suppose first that F does not have modulus 2 or 3. Then we can so determine r, s, t (s and t not both zero) as to make $A' = B' = 0$. Hence we consider (I) for $A = B = 0$. The algebra obtained by applying transformation (1) will have $A' = B' = 0$ if and only if

$$r = -\tfrac{1}{3}t(\alpha + D) = -\tfrac{1}{3}t(\beta + D).$$

Let first $\alpha \neq \beta$. Then must $r = t = 0$. Hence $I = si$, $J = s^2 j$, and

$$b' = s^3 b,\ \beta' = s^2\beta,\ a' = s^3 a,\ \alpha' = s^2\alpha,\ D' = s^2 D,\ \delta' = s^3\delta,\ d' = s^4 d.$$

Now $\delta \neq 0$, in view of the irreducibility of C. Further α and β are not both zero, since $\alpha \neq \beta$. Hence s^3 and s^2 are uniquely determined. *There is at most one transformation converting an algebra* (I) *with* $A = B = 0$, $\alpha \neq \beta$, *into a similar algebra.* Moreover, the conditions for equivalence are obvious.

Let next $\alpha = \beta$. Then for $A = B = 0$, $r = -(\beta + D)t/3$, we have

$$\alpha' = \beta' = \beta s^2 + st(a + b + \delta) + t^2[d + \tfrac{1}{3}(\beta^2 - \beta D + D^2)],$$

$$a' = as^3 + s^2 t[d + \tfrac{1}{3}\beta(\beta + D)] + \tfrac{1}{3}st^2[\delta(\beta + D) + a(D - 2\beta) + b(4\beta - 2D)]$$
$$+ t^3[b\delta + \tfrac{1}{3}d(D - 2\beta) + \tfrac{1}{27}(\beta + D)(\beta - 2D)(2\beta - D)],$$

$$D' = Ds^2 + st(2a + 2b - \delta) + \tfrac{1}{3}t^2(6d + D^2 + 2\beta D - 2\beta^2),$$

$$\delta' = \delta s^3 + \tfrac{2}{3}s^2 t(2\beta - D)^2 + st^2\delta(2\beta - D) + t^3[\delta^2 - \tfrac{2}{27}(2\beta - D)^3],$$

the value of b' being derived from a' by interchanging a with b, while the long expression for d' will not be required. Set $\rho = D - 2\beta$. Then

$$\rho' = \rho s^2 - 3\delta st - \tfrac{1}{3}\rho^2 t^2,\ a' - b' = (a - b)\,C,\ C \equiv s^3 + \rho st^2 - \delta t^3.$$

Since $\delta \neq 0$, we may apply a preliminary transformation and make $\rho \neq 0$. We proceed to determine all the transformations of the algeba into itself. For $t = 0$, the transformation is the identity. Let $t \neq 0$ henceforth. For a non-commutative algebra a very simple treatment is possible. Since $a \neq b$, we have $C = 1$. Hence $\delta' - \delta C = 0$, viz.,

$$s^2\rho^2 - 3st\delta\rho + t^2(3\delta^2 + \rho^3/9) = 0.$$

Combining this equation with $\rho' = \rho$, we get $t\sqrt{R} = \mp 3\rho$, where

$$R = -27\delta^2 - 4\rho^3 = \text{discriminant of } C;\ R \neq 0.$$

Note that, if F is the $GF[p^n]$, R is a square. Indeed, the roots of the irreducible equation $x^3 + \rho x - \delta = 0$ are λ, λ^{p^n}, $\lambda^{p^{2n}}$, where $\lambda^{p^{3n}} = \lambda$. Hence

$$R = P^2, \qquad P \equiv (\lambda - \lambda^{p^n})(\lambda - \lambda^{p^{2n}})(\lambda^{p^n} - \lambda^{p^{2n}}), \qquad P^{p^n} = P.$$

The preceding quadratic equation now gives $(\mp s\sqrt{R} - \tfrac{1}{2}9\delta)^2 = \tfrac{1}{4}R$. The

quantity in parenthesis must equal $\pm\frac{1}{2}\sqrt{R}$, this determination of sign being necessary in view of $C=1$. Hence

$$t=\mp 3\rho/\sqrt{R}, \qquad s=-\tfrac{1}{2}\mp\tfrac{1}{2}9\delta/\sqrt{R}.$$

For this result we proceed to give a proof valid for both commutative and non-commutative algebras. Set $\rho=kl^2$, $\delta=kl^3$, $s=ylt$. Then the equations $\delta'=\delta$ and $\rho'=\rho$ become

$$1=t^3l^3(y^3+\tfrac{2}{3}ky^2-ky+k+\tfrac{2}{27}k^2), \qquad 1=t^2l^2(y^2-3y-\tfrac{1}{3}k).$$

Squaring the first, cubing the second, and eliminating tl, we get a quintic equation $Q=0$ in y. It becomes simpler if we set $k=-9w-\frac{27}{4}$. Thus

$$l=\delta/\rho, \qquad k=\rho^3/\delta^2, \qquad w=R/36\delta^2.$$

Hence $w\neq 0$. Removing the factor $3w$ from $Q=0$, we get

$$4y^5-3(4w+7)y^4+(32w+36)y^3+3(w+\tfrac{3}{4})(8w-6)y^2$$
$$-36(w+\tfrac{3}{4})^2y-3(w+\tfrac{3}{4})^2(4w-9)=0.$$

The left member is seen to factor. Hence

$$4UV=0, \quad U\equiv y^2-3y-w+\tfrac{9}{4}, \quad V\equiv y^3-3(w+\tfrac{3}{4})y^2+3(w+\tfrac{3}{4})^2.$$

Since the cubic function C is irreducible in the field F, the same is true of V. Indeed, if we make the above substitutions in C, using capital Y to avoid confusion, we get

$$C=l^3t^3[\,Y^3-9(w+\tfrac{3}{4})\,Y+9(w+\tfrac{3}{4})\,].$$

For $Y=3(w+\frac{3}{4})/y$, the quantity in brackets becomes $9(w+\frac{3}{4})V/y^3$. Hence $V\neq 0$. Then $U=0$ gives $y=\frac{3}{2}\pm\frac{1}{6}\sqrt{R}/\delta$. From the two above equations whose first members are unity, we get t uniquely. The resulting values of s and t are those given above.

The two transformations (1) given by these two sets of values of s and t are inverse operators. Hence an algebra invariant under one is invariant under the other. We find that the conditions that β' shall equal β for *both* sets of values of s and t are equivalent to the following:

$$(3) \qquad (a+b)(D-2\beta)+\delta(D+\beta)=0, \qquad 3d=\beta^2-D^2.$$

But these relations are equivalent to relations (2) for $B=0$. Hence the family of algebras (I) with $A=B=0$, $\alpha=\beta$, and satisfying conditions (3), is invariant under every transformation (1), so that relations (3) hold true when written in the accented letters. Hence from the fact that the two sets of values of s and t satisfy the relations $\beta'=\beta$, $\delta'=\delta$, $D'-2\beta'=D-2\beta$, $a'-b'=a-b$, it follows without computation that they satisfy relations $D'=D$, $d'=d$, $a'=a$, $b'=b$. We have now proved the

THEOREM. *Let F be any field not having modulus 2 or 3. Consider the algebras* (I) *for which the cubic function C is irreducible in F. By a preliminary transformation of units we may make $A = B = 0$. Then the only algebras which admit more than one transformation (and hence three transformations) into themselves are the algebras with $\alpha = \beta$ and satisfying conditions** (3) *and having the discriminant R of C a square in F. When F is a finite field, R is always a square.*

It remains to treat the case when F is the $GF[3^n]$. Then every mark is a cube. Applying a preliminary transformation, we may set $B = 0$. Since $x^3 - b - \beta x$ shall be irreducible, β is not zero. In view of B', every transformation of the algebra into itself must have $t(\alpha + D) = 0$.

Let first $\alpha + D \neq 0$. Then $t = 0$. Hence $\beta' = s^2\beta$, $b' = r^3 - rs^2\beta + s^3 b$. For a transformation of the algebra into itself, $s^2 = 1$, $b = r^3 - r\beta + sb$. The case $s = -1$ is excluded since $r^3 - r\beta + b$ is irreducible. Hence $s = 1$, $r^3 = r\beta$. It was shown above that the discriminant $-27b^2 + 4\beta^3$ of an irreducible cubic $x^3 - \beta x - b$ is a square. Hence β is a square in the $GF[3^n]$. Now the algebra is transformed into itself by $I = i + \beta^{\frac{1}{2}}$ if and only if

$$A = 0,\ \alpha = \beta,\ D = \beta,\ a + b + \delta = 0.$$

There remains the case $B = 0$, $\alpha + D = 0$. Then $B' \equiv 0$, $\alpha' + D' \equiv 0$. By a preliminary transformation we can make also $A = 0$. In view of A', every transformation of the algebra into itself must have $t(\beta + D) = 0$. Since C shall be irreducible, $D - \alpha - \beta \neq 0$. Hence $t = 0$. As above, $s = 1$, $r^3 = r\beta$, $\alpha = \beta$. Hence $D - \alpha - \beta = 0$, so that this case is excluded.

Noting that for modulus 3 the conditions on the algebra cited in the above theorem require that $\alpha + D \neq 0$ in view of the irreducibility of C, we may state that the theorem holds true when F is the $GF[3^n]$.

THEOREM. *When F is the $GF[p^n]$, $p > 2$, all the algebras* (I_0) *for which $x^3 - b - \beta x - Bx^2$ is irreducible in F are equivalent.*

For this algebra we have

$$C = s^3 + 2s^2tB - 4\beta st^2 + 8bt^3.$$

We prove that C vanishes only when $s = t = 0$. For $t \neq 0$, set $s = -2\sigma t$. Then $C = -8t^3(\sigma^3 - b - \beta\sigma - B\sigma^2)$. Hence $C \neq 0$. In view of our earlier results, the family of algebras (I_0) is invariant under the $p^n(p^{2n} - 1)$ transformations (1) with s and t not both zero; while each algebra is transformed into itself by exactly three transformations. The theorem will now follow if we show that there are exactly $\frac{1}{3}p^n(p^{2n} - 1)$ irreducible cubic functions in the

* That these conditions are necessary may be shown without computation. For $\beta \neq 0$, $D \neq 0$, there is more than one transformation only when the quadratic functions β' and D' are identical, apart from a constant factor, the conditions for which are (3). Otherwise $\beta'/D' = \beta/D$ would uniquely determine s/t. The same argument applies to the cases $\beta = 0$, $\beta' \neq 0$; $D = 0$, $D' \neq 0$. When $D \equiv 0$ we have a special case of (3). The case $\beta' \equiv 0$ requires separate treatment.

$GF[p^n]$. But we readily enumerate the reducible cubic functions. Of those with three linear factors, p^n have three equal factors, $p^n(p^n-1)$ have just two equal factors, and $\frac{1}{6}p^n(p^n-1)(p^n-2)$ have distinct factors. Finally, there are $\frac{1}{2}p^{2n}(p^n-1)$ cubic functions with a linear and an irreducible quadratic factor.

The theorem is readily extended to other sets of commutative algebras satisfying relations (2). In particular, it is true for (I_1).

As an illustration of the use of the above transformation theory, we determine the non-equivalent algebras in the $GF[3]$ such that division is always uniquely possible. As above, we may set $B=0$. Then $x^3-b-\beta x$ must be irreducible, so that $b^2=1$, $\beta=1$. By the transformation which merely changes the sign of i, β is unaltered, while b is replaced by $-b$. Hence we may set $b=\beta=1$. Now $\phi\equiv x^3-a-\alpha x-Ax^2$ and $C\equiv s^3+s^2tA+st^2(D-\alpha-1)-t^3\delta$ must be irreducible. Under the transformation $I=i-1$, $J=j+i+1$, the algebra becomes a similar algebra with

$$\alpha'=a+\alpha+A-1, \qquad \alpha'=\alpha-A, \qquad A'=A,$$

$$D'=A+D, \qquad \delta'=\alpha+\delta-A-D, \qquad d'=a+\alpha+A+d+\delta+D-1.$$

Let first $A=0$. Then $a^2=1$, $\alpha=1$ by ϕ; $\delta^2=1$, $-D+\alpha+1=1$ by C, whence $D=1$. The transformed algebra differs from the original only in d' and $d'=d+a+\delta+1$. If $a+\delta+1\neq 0$, we can make d' arbitrary (by a repetition of the transformation) and hence make $d=\delta$. Then $(j-\delta i-D)j=0$. Hence $a+\delta+1=0$, so that $a=\delta=1$, $d\neq 1$. For $d=0$, the algebra is the $GF[3^3]$. For $d=-1$, it is a case of algebra (I_{11}).

Let next $A\neq 0$. By a repetition of the above transformation, we can make $\alpha=0$. Then $a+A=0$ by ϕ; either $D=0$, $A\neq\delta$, or $D\neq 0$, $A=\delta$, by C. For the first case, $\alpha=D=0$, $a=\delta$, $A=-\delta$, $\delta\neq 0$, $d\neq\delta$; then if $d=0$, Δ_r vanishes for $r=t=1$, $s=0$; if $d=-\delta$, the first row of Δ_r is δ times the third row for $r=\delta$, $s=\delta+1$, $t=1$. Hence must $D\neq 0$, $A=\delta$, $a=-\delta$, $\alpha=0$, $\delta\neq 0$, $d\neq\delta$. Then Δ_r for $t=1$, $s=0$, becomes $r^2D-rd-1$. Hence either $D=-1$, $d=0$, or else $D=1$, $d\neq 0$, whence $d=-\delta$. In either case Δ_r vanishes only when $r=s=t=0$. For example, if $t=1$, $s=-1$, $\Delta_r=\mp(r^2-\delta r-1)$. The resulting algebras are*

$$i^2=j,\ ij=1+i,\ ji=\mp 1\pm j,\ j^2=\pm i-j;$$

$$i^2=j,\ ij=1+i,\ ji=\mp 1\pm j,\ j^2=\mp 1\pm i+j.$$

No two of these algebras are equivalent under a transformation (1). Since $B=\alpha=0$, $D\neq 0$, $B'=tD$. Hence $B'=0$ requires that $t=0$. Then $s\neq 0$, $\alpha'=rsA$. Hence $\alpha'=0$ requires that $r=0$. But the transformation which merely changes the sign of i does not preserve the relation $ij=1+i$.

* If in Δ for the field $i^2=j$, $ij=1+i$, $j^2=i+j$, we replace r and s by $r-t$ and $s+t$; r by $r+t$; s and t by $s+t$ and $-t$; t by $-t$, we obtain the determinant Δ_r of the four algebras, respectively.

THEOREM. *When F is the $GF[3]$ there are exactly six non-equivalent algebras in which division is always uniquely possible. The two commutative algebras fall under* (I_0) *and* (I_1).

Finally, I give a representative of each of the 36 non-equivalent sets of non-commutative algebras in the $GF[5]$ in which division is always uniquely possible. The two existing types of commutative algebras are given by (I_0) and (I_1), and are not listed. The tables relate to formula (I) with $A = B = 0$.

$b=2, \beta=1$

α	a	d	δ	D
1	-2	0	± 1	-1
-1	1	1	1	1
-1	1	0	± 1	2
-1	-1	1	2	-1
-1	-1	0	± 2	-2
2	± 2	-1	± 1	0
2	∓ 2	2	± 2	1
-2	± 1	1	1	0
-2	± 1	-2	± 2	2
-2	± 1	-2	-2	-2

$b=2, \beta=2$

α	a	d	δ	D
1	± 2	-1	1	0
1	∓ 2	2	± 1	-1
1	± 2	2	-2	1
-1	∓ 1	1	± 2	0
-1	± 1	-2	∓ 1	2
-1	± 1	-2	-1	-2
-2	1	-1	1	2
-2	1	0	± 2	-1
-2	-1	-1	2	-2
-2	-1	0	± 1	1

§ 3. *Derivation of an mk-tuple algebra from an m-tuple algebra.*

From an m-tuple algebra in a general field F, we readily deduce an mk-tuple algebra. We take as F the field $f(\rho)$ obtained from the field f by the adjunction of a root ρ of an equation of degree k, irreducible in f. Then if $e_1, \cdots, e_m$ are the units of the given algebra in F, we take the products $e_r\rho^s (r = 1, \cdots, m;\ s = 0, 1, \cdots, k-1)$ as the mk units of the desired algebra.

By way of illustration, we construct from (I_0) a 6-tuple algebra, which will be employed later. Let F be the field $f(\rho)$, where $\rho^2 = \nu$, ν being a not-square in f. Set $b = a + c\rho$, $\beta = \alpha + \gamma\rho$, $B = A + C\rho$, $\rho i = k$, $\rho j = l$. We obtain the 6-tuple algebra in f:

$$
(\mathrm{II})\quad \begin{cases} \rho^2 = \nu,\ \rho i = k,\ \rho j = l,\ \rho k = \nu i,\ \rho l = \nu j,\ i^2 = j,\ ik = l,\ k^2 = \nu j, \\ ij = a + c\rho + \alpha i + \gamma k + Aj + Cl,\ il = kj = \nu c + a\rho + \nu\gamma i + \alpha k + \nu Cj + Al, \\ j^2 = 4aA + 4\nu cC - \alpha^2 - \nu\gamma^2 + (4aC + 4Ac - 2\alpha\gamma)\rho - 8ai - 8ck - 2\alpha j - 2\gamma l, \\ l^2 = \nu j^2,\ jl = \rho j^2,\ kl = \nu a + \nu c\rho + \nu\alpha i + \nu\gamma k + \nu Aj + \nu Cl; \end{cases}
$$

the value of j^2 to be inserted in l^2 and jl. Similarly, from (I′) we deduce a field (II′) whose multiplication-table differs from (II) only in the values of j^2, l^2, jl. It may be shown that the determinant $\Delta(a)$ of (II′) can be transformed into that of (II). Since the algebras (I_0), and hence the algebras (II), are all equivalent, it suffices to make the proof for the case $A = C = c = \gamma = 0$. With the notation $a_1 + a_2 i + a_3 j + a_4 \rho + a_5 k + a_6 l$ for the general element, the determinant $\Delta(a)$ of (II′) becomes $\Delta(a)$ of (II) upon replacing a_1 by $a_1 + \alpha a_3$, a_4 by $a_4 + \alpha a_6$, a_3 by $-2a_3$, a_6 by $-2a_6$.

§ 4. *Commutative algebras with four units*, $1, i, j, k$.

Let F be a field not having modulus 2. By a transformation of units we may make * $i^2 = j$. Indeed there is some element e such that e^2 is not a linear function of e and we may set $i = e$, $j = e^2$. For, suppose that for every element e, e^2 is of the form $re + s$. Then $i^2 = ai + A$. Let $I = i - a/2$. Hence $I^2 = A + a^2/4$. Proceeding similarly with j we may set $i^2 = \alpha$, $j^2 = \beta$. Let $ij = r_1 + r_2 i + r_3 j + r_4 k$. Then, for arbitrary l,

$$(i + lj)^2 \equiv \alpha + l^2\beta + 2lr_1 + 2lr_2(i + lj) + 2lj(r_3 - lr_2) + 2lr_4 k$$

must be a linear function of $i + lj$. Hence $r_2 = r_3 = r_4 = 0$. Hence

$$i^2 = \alpha, \qquad ij = r_1, \qquad i(r_1 i - \alpha j) = 0.$$

In a 4-tuple algebra with $i^2 = j$, we may set $ij = k$. For, suppose that $ij = a + bi + cj$ with no term in k and that $ik = c_1 + c_2 i + c_3 j + c_4 k$. In

$$(l_1 + l_2 i)(x_1 + x_2 i + x_3 j + x_4 k),$$

the coefficient of k is $(l_1 + l_2 c_4)x_4$ and hence may be made zero by choice of l_1 and l_2, not both zero. The above product could not then be made equal to k, so that division by $l_1 + l_2 i$ would not always be possible.

Consider a 4-tuple algebra of the form

$$\text{(III)} \quad \begin{cases} i^2 = j, \ ik = ki = c_1 + c_2 i + c_3 j + c_4 k, \ j^2 = d_1 + d_2 i + d_3 j + d_4 k, \\ ij = ji = k, \ jk = kj = k_1 + k_2 i + k_3 j + k_4 k, \ k^2 = s_1 + s_2 i + s_3 j + s_4 k. \end{cases}$$

Suppose that $d_1 = 0$. Then the constant term in the product

$$(l_2 i + l_3 j)(x_1 + x_2 i + x_3 j + x_4 k)$$

is $(c_1 l_2 + k_1 l_3)x_4$ and hence may be made zero by choice of l_2 and l_3, not both zero. The product is then never 1, so that $l_2 i + l_3 j$ would have no inverse. Hence $d_1 \neq 0$.

We can now prove † that there is no root in the field F of

$$E \equiv x^4 - d_1 - d_2 x - d_3 x^2 - d_4 x^3 = 0.$$

* The same argument applies to algebras with m units, $m > 2$.

† The analogous theorem on algebras with m units is proved similarly.

Suppose that $E=0$ has in F the root $x=-w$. Let

$$I=i+w, \qquad J=j+2wi+w^2, \qquad K=k+3wj+3w^2i+w^3.$$

Then $I^2=J$, $IJ=K$, while the constant term in J^2, expressed as a linear function of I, J, K, is $d_1-d_2w+d_3w^2-d_4w^3-w^4$, and hence is zero. Hence the algebra is equivalent to one of the form (III) with $d_1=0$, contrary to the above result.

Next, the function E is not the product of two quadratic factors with coefficients in F. Indeed, if $E=(x^2-px-q)(x^2-rx-s)$, then

$$(j-pi-q)(j-ri-s)=0.$$

The preceding results lead to the following

THEOREM. *In a field F not having modulus 2, a quaternary linear algebra in which multiplication is commutative and division is always uniquely possible is equivalent in F to an algebra of the form* (III) *in which $E\equiv x^4-d_1-d_2x-d_3x^2-d_4x^3$ is irreducible in F.*

For algebra (III) the determinant $\Delta(l)$ is

$$(4)\qquad \begin{vmatrix} l_1 & c_1l_4 & d_1l_3+k_1l_4 & c_1l_2+k_1l_3+s_1l_4 \\ l_2 & l_1+c_2l_4 & d_2l_3+k_2l_4 & c_2l_2+k_2l_3+s_2l_4 \\ l_3 & l_2+c_3l_4 & l_1+d_3l_3+k_3l_4 & c_3l_2+k_3l_3+s_3l_4 \\ l_4 & l_3+c_4l_4 & l_2+d_4l_3+k_4l_4 & l_1+c_4l_2+k_4l_3+s_4l_4 \end{vmatrix}.$$

The problem is to determine c_i, d_i, k_i, s_i $(i=1,2,3,4)$ in F so that $\Delta(l)$ vanishes only when each l_i vanishes. We apply the principle * that any algebra (III) is equivalent to one of like form in which the d_i are given marks for which E is irreducible in F.

First, let F be the field of order 3. Since x^4-x-1 is irreducible modulo 3, we set $d_1=d_2=1$, $d_3=d_4=0$. For $l_3=l_4=0$, (4) becomes

$$l_1^4+c_4l_1^3l_2-c_3l_1^2l_2^2+c_2l_1l_2^3-c_1l_2^4.$$

Since it shall vanish only when $l_1=l_2=0$, the c_i must satisfy one of the four sets of conditions in the $GF[3]$:

$$(5)\qquad \begin{cases} c_1=1,\ c_3=0,\ c_2+c_4\neq 0; & c_1=1,\ c_3\neq 0,\ c_2+c_4=0; \\ c_1=-1,\ c_3=-1,\ c_2+c_4\neq 0; & c_1=-1,\ c_3\neq -1,\ c_2+c_4=0. \end{cases}$$

For $l_4=0$, $l_3\neq 0$, we may set $l_1=xl_3$, $l_2=yl_3$, and remove the factor l_3^4 from (4). In the resulting function we give to x, y in turn the 9 pairs of values 0,

* See p. 30 of my memoir, cited above. Compare next to the last theorem of § 2; also the verification of the principle for algebra (IV) below.

$0; 1, 0; \cdots; -1, -1$. Hence no one of the nine resulting expressions is to vanish:

$$(6)\quad \begin{cases} k_2-k_1, \quad k_3-k_1, \quad c_1+c_2-c_3+c_4+k_1+k_2-k_3+k_4, \\ \qquad\qquad -k_3-k_1, \quad -c_2-c_3-c_4+k_2+k_3+k_4, \\ c_2-c_3+k_2-k_3, \quad -1-c_1-c_2+c_4+k_1+k_2-k_4, \\ 1-c_1-c_2-c_4-k_1-k_2-k_4, \quad c_1+c_2-c_3-k_1-k_2+k_3. \end{cases}$$

For each set of c's given by (5), one readily determines the sets of k's satisfying (6). For example, if $c_1=1$, $c_2=0$, $c_3=0$, $c_4=1$, we must have $k_1=1$, $k_2=-1$, $k_3=0$, $k_4=0$. It remains to examine (4) for $l_4 \neq 0$. In view of the homogeneity, we may set $l_4=1$. For $(l_1, l_2)=(1, -1)$ and $(0, 0)$, (4) becomes respectively

$$(l_3-s_2)(l_3^2+1)-s_3(l_3^2+l_3-1)-s_4, \quad (l_3-s_2)(l_3^2+l_3)-s_3(l_3-1)+s_4(l_3^2-l_3).$$

Neither is to vanish for any value of l_3. Hence no one of the expressions

$$s_2-s_3+s_4, \quad 1+s_2+s_3-s_4, \quad -1+s_2-s_3-s_4, \quad s_3, \quad s_2-1, \quad s_3+s_4$$

is to vanish. Hence $s_2=0$, $s_3=1$, $s_4=0$. Then (4) vanishes for $l_1=0$, $l_2=1$, $l_3=-1$. In the same manner are excluded all cases in which $c_1=1$, $c_3=0$, except the $GF[3^4]$. If $c_1=1$, $c_2=0$, $c_3=1$, $c_4=0$, then $(k_1, k_2, k_3, k_4)=(0, 1, -1, 0)$, $(1, 0, 0, 0)$, or $(-1, 0, 0, -1)$; the first is excluded, while for the second we must have $s_1=1$, $s_2=1$, $s_3=0$, $s_4=-1$, and therefore

$$(\text{IV})\quad \begin{gathered} i^2=j, \quad ij=ji=k, \quad ik=ki=1+j, \quad jk=kj=1, \\ j^2=1+i, \quad k^2=1+i-k. \end{gathered}$$

A computation showed that, for algebra (IV), $\Delta(l)$ vanishes only when every $l_i=0$. An immediate proof will be given below. In (IV), set

$$(7)\qquad I=\alpha+\beta i+\gamma j+\delta k, \quad J=I^2, \quad K=IJ.$$

Then condition that $1, I, J, K$ shall be independent units reduces to

$$\begin{vmatrix} \beta & \gamma & \delta \\ \delta^2+\gamma^2 & \beta^2-\beta\delta & -\delta^2-\beta\gamma \\ -\delta^3 & \delta^3-\beta^2\gamma & \beta^3-\beta^2\delta+\gamma^3+\delta^3 \end{vmatrix} \not\equiv 0 \pmod 3.$$

But this determinant vanishes only when $\beta \equiv \gamma \equiv -\delta \pmod 3$. Hence there are exactly $3(27-3)=72$ valid transformations (7). Exactly four of these transform algebra (IV) into itself, viz.,

$$T: \quad I=i-j, \quad J=1+i+j+k, \quad K=-1+i,$$

and T^2, T^3, $T^4 =$ identity. Hence there are exactly 18 algebras with $I^2 = J$ and $IJ = K$, which are equivalent to (IV). To obtain them, we apply transformation (7) with $(\beta, \gamma, \delta) = (0, 0, \pm 1)$, $(\pm 1, 0, 0)$, $(0, 1, 1)$, $(1, 1, 0)$, and replace α by $\alpha \pm 1$ in the first case, α by $\alpha + 1$ in the third. The resulting algebras are

IK	J^2	JK	K^2
$1-\alpha I+J+\alpha K$	$\alpha^2 \mp \alpha - 1 \pm I - J + \alpha K$	$\alpha \pm 1 - \alpha^2 I + (\alpha^2 - 1)K$	$\alpha^2 - 1 - (\alpha \mp 1)I - J - (\alpha \pm 1)K$
$1-\alpha I+J+\alpha K$	$-\alpha^2 \mp \alpha + 1 + (\alpha \pm 1)I + \alpha K$	$\pm 1 - \alpha^2 I + \alpha^2 K$	$1 - \alpha^2 \pm I - (\alpha \pm 1)K$
$-1-\alpha^2+\alpha I+\alpha K$	$\alpha^2 + 1 - J + \alpha K$	$I + \alpha J + (\alpha^2 - 1)K$	$1 + \alpha I + J - \alpha K$
$-1-\alpha^2+\alpha I+\alpha K$	$1 - \alpha I + J + \alpha K$	$\alpha + I + \alpha J + (\alpha^2 + 1)K$	$-1 + \alpha I + J - \alpha K$

If we add $-x^4$ to the various functions in the second column, after replacing I, J, K by x, x^2, x^3, respectively, we obtain the 18 existing irreducible functions of the fourth degree modulo 3. We have therefore verified the principle cited above.

Special attention should be given to the algebra in the third line of the table for the case $\alpha = 0$, viz.,

(V) $I^2 = J$, $IJ = K$, $J^2 = 1 - J$, $IK = -1$, $JK = I - K$, $K^2 = 1 + J$.

Its determinant $\Delta(l)$ equals $F_{1,3} + F_{2,4}$, where

$$F_{x,y} \equiv x^4 - 2x^3y - x^2y^2 + 2xy^3 + y^4.$$

Since $F_{x,y} \equiv (x^2 + y^2)^2 \pmod 3$, $\Delta(l)$ vanishes only when each $l_i = 0$.

Next, let F be the field of order 5. Since $x^4 - 2$ is irreducible modulo 5, we set $d_1 = 2$, $d_2 = d_3 = d_4 = 0$ in (III). Then, for $l_2 = l_4 = 0$, (4) equals

$$(l_1^2 - 2l_3^2)(l_1^2 + k_4 l_1 l_3 - k_2 l_3^2).$$

Hence the second factor must be irreducible modulo 5, so that $k_4^2 - k_2$ must be a not-square, viz., 2 or -2. Thus

(8) $(k_4, k_2) = (0, \pm 2)$, $(\pm 1, 3)$, $(\pm 1, 4)$, $(\pm 2, 1)$, or $(\pm 2, 2)$.

If we set $I = 2i$, $J = -j$, $K = -2k$, we get

$$I^2 = J, \quad IJ = K, \quad J^2 = 2, \quad IK = c_1 - 2c_2 I - c_3 J + 2c_4 K,$$
$$JK = 2k_1 + k_2 I - 2k_3 J - k_4 K.$$

Hence we may change the sign of k_4 without altering k_2. Hence we need retain only the upper signs in the last four cases of (8). Consider the case $k_4 = 0$, $k_2 = 2$. By repetitions of the above transformation we can multiply c_4 (or c_2) by any power of 2; hence there are three sub-cases: $c_4 = 1$; $c_4 = 0$, $c_2 = 0$; $c_4 = 0$, $c_2 = 1$. By an investigation entirely analogous to that employed above for the case modulus 3, but here much more laborious, I found that the cases

$c_1 = 0$ and $c_1 = \pm 1$ are to be excluded, while for $c_1 = 2$ the only algebra is the $GF[5^4]$. For $c_1 = -2$, I examined only one of the three sub-cases cited above, viz., that for which $c_4 = 0$, $c_2 = 0$, and found that we must have $c_3 = 0$, $k_1 = k_3 = 0$, $s_1 = s_2 = s_4 = 0$, $s_3 = 2$. The resulting algebra is

$$\text{(VI)} \qquad i^2 = j,\ ij = k,\ j^2 = 2,\ ik = -2,\ jk = 2i,\ k^2 = 2j.$$

For it $\Delta(l)$ is seen to equal $(l_1^2 - 2l_3^2)^2 + 2(l_2^2 - 2l_4^2)^2$. Since 2 and -2 are not squares, $\Delta(l)$ vanishes only when each l_i vanishes.

Let F be any field for which -1 is a square and ν is a not-square. Replacing 2 by ν in (VI) we obtain the algebra

$$\text{(VII)} \qquad i^2 = j,\ ij = k,\ j^2 = \nu,\ ik = -\nu,\ jk = \nu i,\ k^2 = \nu j,$$
$$\Delta(l) = (l_1^2 - \nu l_3^2)^2 + \nu(l_2^2 - \nu l_4^2)^2.$$

Seeking a generalization of algebras (V) and (VII), we set

$$\text{(III}') \quad i^2 = j,\ ij = k,\ j^2 = d_1 + d_3 j,\ ik = c_1,\ jk = k_2 i + k_4 k,\ k^2 = s_1 + s_3 j.$$

The simplicity of our proofs that division is always uniquely possible in algebras (V) and (VII) is due to the fact that for each $\Delta(l)$ equals the sum of a function of l_1, l_3 and a function of l_2, l_4. We now require that (III′) shall have the same property.* The conditions for this are found to be

$$s_1 + s_3 k_4 = 0,\ k_2 + 2c_1 + s_3 = 0,\ 2s_1 - 2c_1 d_3 = 0,\ d_1 + k_2 + 2c_1 = 0,$$
$$2c_1 k_2 + s_3 k_2 + s_3 d_1 - s_1 d_3 + s_1 k_4 = 0,\ k_2 c_1 d_3 - k_2 s_1 = 0,$$
$$k_4 d_1 + c_1 d_3 = 0,\ d_3 s_1 - k_4 c_1 d_3 - s_3 d_1 - 2k_2 c_1 - k_2 d_1 = 0.$$

The coefficient $-s_3 c_1 k_2$ of l_4^4 in $\Delta(l)$ cannot vanish. Hence the above conditions are satisfied if and only if

* Note added May 25, 1906. It may be shown that, if the general algebra (III) has this property, d_2, d_4, k_1, k_3, s_2 and s_4 all vanish, so that (III) reduces to (III′). As at the beginning of § 5, we must have $c_2 = c_3 = c_4 = 0$, $c_1 \neq 0$. Also $s_4 = -k_3$, since the coefficient of $l_1^3 l_4$ in (4) must vanish. I expanded determinant (4) with these values inserted. Since the coefficients of $l_2^3 l_3$, $l_1^2 l_2 l_3$, $l_1 l_2^2 l_4$, $l_1 l_2^2 l_3$, $l_1^2 l_2 l_4$, $l_2 l_3^3$, $l_1 l_2 l_4^2$, $l_2^2 l_3 l_4$, $l_1^2 l_3 l_4$, $l_1 l_4^3$, $l_1 l_2 l_3 l_4$ must vanish, we have respectively,

$$k_1 = -c_1 d_4, \quad k_3 = -d_2, \quad s_2 = c_1 d_4, \quad k_2 = -d_1 - 2c_1, \quad s_3 = d_1,$$
$$d_1 d_2 - c_1 d_2 - c_1 d_3 d_4 = 0, \quad d_1 d_2 + 3c_1 d_2 + c_1 d_4 k_4 = 0, \quad d_1 d_2 - c_1 d_2 - s_1 d_4 = 0,$$
$$c_1 d_4 + d_2 d_3 - d_1 d_4 + 0, \quad c_1^2 d_4 - c_1 d_1 d_4 + s_1 d_2 = 0, \quad 2s_1 + c_1 d_4^2 - d_2^2 - 2c_1 d_3 = 0.$$

Since the terms involving l_1 and l_2 alone are $l_1^4 - c_1 l_2^4$, c_1 must be a not-square.

The conditions arising from other coefficients are not needed for the argument. From the two conditions preceding the last, we derive $s_1 d_2 = c_1 d_2 d_3$. If $d_2 \neq 0$, the last condition now becomes $c_1 d_4^2 - d_2^2 = 0$, which is impossible, c_1 being a not-square. Hence $d_2 = 0$. The above conditions are seen to require that $d_4 = 0$.

$s_1 = c_1 d_3,\ s_3 = d_1,\ k_2 = -d_1 - 2c_1,\ k_4 = -c_1 d_3 d_1^{-1},\ (4d_1 + d_3^2)(c_1 + d_1) = 0.$

The two sets of solutions are

$$d_1 = -c_1,\ s_3 = -c_1,\ s_1 = c_1 d_3,\ k_2 = -c_1,\ k_4 = d_3; \tag{9}$$

$$d_1 = s_3 = -\tfrac{1}{4}d_3^2,\ s_1 = c_1 d_3,\ k_2 = \tfrac{1}{4}d_3^2 - 2c_1,\ k_4 = 4c_1 d_3^{-1}. \tag{9'}$$

For these respective sets of values

$$\Delta(l) = F_{l_1, l_3} - c_1 F_{l_2, l_4}, \qquad F_{x,y} \equiv (x^2 + d_3 xy + c_1 y^2)^2; \tag{10}$$

$$F_{x,y} \equiv [x + \tfrac{1}{2}d_3 y]^2 [x^2 + 4c_1 d_3^{-1} xy + (2c_1 - \tfrac{1}{4}d_3^2) y^2]. \tag{10'}$$

In the latter case $\Delta(l)$ vanishes when $l_1 = l_2 = d_3$, $l_3 = l_4 = -2$. Hence the set (9′) is excluded. In order that $\Delta(l)$, given by (10), shall vanish only when each l_i vanishes, it is necessary and sufficient that c_1 be a not-square and that $z^2 + d_3 z + c_1$ be irreducible in the field F. We obtain the algebra

$$\text{(VIII)} \quad \begin{cases} i^2 = j,\ ij = k,\ ik = c,\ j^2 = -c + dj,\ jk = -ci + dk,\ k^2 = cd - cj, \\ c \text{ and } d^2 - 4c \text{ each a not-square in the field } F. \end{cases}$$

Note that $kj = (ij)j = i(j^2)$, $k^2 = i(jk)$; also that the sub-algebra defined by the units 1 and j is a field. When F is the Galois field of order p^n, there exists an algebra (VIII). For, take as c any fixed not-square; then when d ranges over the p^n marks, $d^2 - 4c$ takes $1 + \frac{1}{2}(p^n - 1)$ distinct values each not zero. But only $\frac{1}{2}(p^n - 1)$ of the marks are squares. Hence there is some value of d for which $d^2 - 4c$ is a not-square.

A noteworthy generalization of algebra (VIII) may be obtained by making the transformation

$$I = i + t, \qquad J = j + 2it + t^2, \qquad K = k + 3jt + 3it^2 + t^3.$$

Simplifications arise if we set $\delta = d - 2t^2$, $\gamma = c - t^2 d + t^4$. Then

$$\text{(IX)} \quad \begin{cases} I^2 = J, \qquad IJ = K, \qquad IK = \gamma + t^2\delta + 4t^3 I - 6t^2 J + 4tK, \\ J^2 = -\gamma - 2t\delta I + (\delta - 4t^2) J + 4tK, \\ JK = 2\delta t^3 + (8t^4 - 4t^2\delta - \gamma) I - 20t^3 J + (\delta + 12t^2) K, \\ K^2 = \gamma\delta + t^2\delta^2 + 4t^2\gamma + 8t^4\delta + (32t^5 - 4t^3\delta - 4t\gamma) I \\ \qquad\qquad + (-\gamma - 10t^2\delta - 64t^4) J + (6t\delta + 32t^3) K, \\ \delta^2 - 4\gamma \text{ and } \gamma + t^2\delta + t^4 \text{ each a not-square in the field.} \end{cases}$$

§ 5. *Commutative algebras with 6 units.*

We seek an algebra such that, if the general element be given the notation $r_1 + r_2 i + r_3 j + r_4 k + r_5 l + r_6 m$, the determinant $\Delta(r)$ shall equal the sum of a function of r_1, r_3, r_5 and a function of r_2, r_4, r_6. Set

$$(11)\quad i^2=j,\ ij=k,\ ik=l,\ il=m,\ im=c_1+c_2i+c_3j+c_4k+c_5l+c_6m.$$

For $r_3=r_4=r_5=r_6=0$, $\Delta(r)$ equals

$$r_1^6+r_1^5r_2c_6-r_1^4r_2^2c_5+r_1^3r_2^3c_4-r_1^2r_2^4c_3+r_1r_2^5c_2-r_2^6c_1.$$

Hence must $c_2=c_3=c_4=c_5=c_6=0$, $c_1\neq 0$.

We are seeking the algebra in 6 units which corresponds to algebra (VIII) in 4 units. For the latter, the first and third units, 1 and j, form a sub-algebra, while the rest of the multiplication-table of the units may be deduced by the associative law, viz., $jk=i(j^2)$, $k^2=i(jk)$. For the 6-tuple algebra we therefore assume that the units $1, j, l$ form a sub-algebra and that

$$(12)\quad jk=i(j^2),\ k^2=i(jk),\ lm=i(l^2),\ m^2=i(lm),\ kl=jm=i(jl),\ km=i(jm).$$

Hence the algebra must be of the form *

$$(\mathrm{X})\quad \begin{cases} i^2=j,\ ij=k,\ ik=l,\ il=m,\ im=c, \\ j^2=d_1+d_3j+d_5l, \qquad l^2=m_1+m_3j+m_5l, \qquad jl=f_1+f_3j+f_5l, \\ jk=d_1i+d_3k+d_5m,\ lm=m_1i+m_3k+m_5m,\ kl=jm=f_1i+f_3k+f_5m, \\ k^2=d_1j+d_3l+d_5c,\ m^2=m_1j+m_3l+m_5c,\ km=f_1j+f_3l+f_5c, \end{cases}$$

it being understood that multiplication is commutative, $ij=ji$, etc.

The determinant $\Delta(r)$ of (X) with certain rows and columns interchanged is

$$(13)\quad \begin{vmatrix} r_1 & A_1 & B_1 & r_6c & E_3c & D_3c \\ r_3 & A_2 & B_2 & r_2 & E_1 & D_1 \\ r_5 & A_3 & B_3 & r_4 & E_2 & D_2 \\ r_2 & E_1 & D_1 & r_1 & A_1 & B_1 \\ r_4 & E_2 & D_2 & r_3 & A_2 & B_2 \\ r_6 & E_3 & D_3 & r_5 & A_3 & B_3 \end{vmatrix},$$

$$A_1=r_3d_1+r_5f_1,\ B_1=r_3f_1+r_5m_1,\ E_1=r_4d_1+r_6f_1,\ D_1=r_4f_1+r_6m_1$$

$$A_2=r_1+r_3d_3+r_5f_3,\ B_2=r_3f_3+r_5m_3,\ E_2=r_2+r_4d_3+r_6f_3,\ D_2=r_4f_3+r_6m_3,$$

$$A_3=r_3d_5+r_5f_5,\ B_3=r_1+r_3f_5+r_5m_5,\ E_3=r_4d_5+r_6f_5,\ D_3=r_2+r_4f_5+r_6m_5.$$

We desire that (13) shall equal

* I was first led to this algebra by postulating (11), giving to $1, i, j, k, l, m$ the weights 0, 1, 2, 3, 4, 5; assuming that, if the sum of the weights of two units is even (odd) their product is a linear function of the units of even (odd) weight; finally requiring that the matrix of the determinant $\Delta(r)$ shall take the form (13), viz., $\begin{pmatrix} A & B' \\ B & A \end{pmatrix}$, where the rows of matrix B' form a cyclic permutation of the rows of B, aside from the factor c. These properties correspond to those of algebra (VIII).

$$(14)\qquad \begin{vmatrix} r_1 & A_1 & B_1 \\ r_3 & A_2 & B_2 \\ r_5 & A_3 & B_3 \end{vmatrix}^2 - c \begin{vmatrix} r_2 & E_1 & D_1 \\ r_4 & E_2 & D_2 \\ r_6 & E_3 & D_3 \end{vmatrix}^2,$$

and hence that the remaining terms in the Laplace development of (13) shall vanish identically. The conditions for this all follow from

$$m_1 = -cf_5,\ m_3 = cd_5,\ f_1 = -c - cd_5,\ f_3 = -d_1,\ d_1^2 - cd_3d_5 - cd_3 + cf_5 = 0,$$

$$c + cd_5 + cd_5^2 + d_1f_5 = 0,\ f_1m_5 + cf_5^2 - cd_1 = 0,\ d_1m_5 + cd_3 + cd_5f_5.$$

From these we derive

$$cm_5 = (-cd_5 - cd_5^2 - d_1f_5)m_5 = cd_5(f_5^2 - m_5 - m_5d_5) + f_5(-d_1m_5 - cd_5f_5),$$

and hence $cm_5 = cd_5d_1 + cf_5d_3$. But $c \neq 0$. We may now readily verify that the above set of conditions is equivalent to the following:

$$(15)\qquad \begin{cases} m_1 = -cf_5,\ m_3 = cd_5,\ m_5 = d_1d_5 + d_3f_5, \\ f_1 = -c - cd_5,\ f_3 = -d_1,\ f_5 = d_3 + d_2d_5 - c^{-1}d_1^2, \end{cases}$$

$$(16)\qquad c^2 + c^2d_5 + c^2d_5^2 + cd_1d_3 + cd_1d_3d_5 - d_1^3 = 0.$$

It remains to impose the condition that (14) shall vanish only when every r_i vanishes. Now the second determinant in (14) is the same function of r_2, r_4, r_6 that the first is of r_1, r_3, r_5. Hence we must take c to be a not-square and require that

$$(17)\qquad \begin{vmatrix} r_1 & A_1 & B_1 \\ r_3 & A_2 & B_2 \\ r_5 & A_3 & B_3 \end{vmatrix}$$

shall vanish only when r_1, r_3, r_5 all vanish. Now (17) is the determinant $\Delta(r)$ of the subalgebra $(1, j, l)$. We proceed to determine the conditions under which the latter is a field. Now d_5 is not zero.* For, if $d_5 = 0$, then $m_3 = 0$, and the elements of the fifth row of (13) would all vanish for $r_3 = r_4 = 0$, $r_1 = -r_5f_3$, $r_2 = -r_6f_3$. Hence we may take as the units of the subalgebra $1, j, L = d_1 + d_3j + d_5l$. Then

$$(18)\qquad \begin{cases} j^2 = L, \qquad jL = (f_5 + d_3)L + (d_1 + d_5f_3 - d_3f_5)j + d_5f_1 - f_5d_1, \\ L^2 = (d_3^2 + d_5m_5 + 2d_1 + 2d_3f_5)L + (d_3^2m_3 + 2d_3d_5f_3 - d_3d_5m_5 - 2d_3^2f_5)j \\ \qquad\qquad + d_5^2m_1 + 2d_3d_5f_1 - d_1^2 - d_1d_5m_5 - 2d_1d_3f_5. \end{cases}$$

* We do not retain the condition $d_5 \neq 0$ in the final theorem. For, if $d_5 = 0$, we multiply (19) by d_1^3 and set $z = d_1y$. After eliminating d_3, we get $(z + c^2)^2(z - d_1^3) = 0$. Hence the cubic is reducible if $d_5 = 0$.

The three conditions for the identity $j(jL) = L^2$ are seen to follow from relations (15) and (16). Hence the subalgebra (18) is a field if

$$x^3 - (f_5 + d_3)x^2 - (d_1 + d_5 f_3 - d_3 f_5)x + f_5 d_1 - d_5 f_1 = 0$$

is irreducible in F. Multiplying the left member by c^3, setting $y = cx$, and applying (15), we get

$$(19) \quad y^3 - (2cd_3 + cd_3 d_5 - d_1^2)y^2 - (cd_1 - cd_1 d_5 - cd_3^2 - cd_3^2 d_5 + d_1^2 d_3)cy - c^4 = 0.$$

THEOREM. *For any not-square c and any marks d_1, d_2, d_3 such that the cubic* (19) *is irreducible in the field F, and such that condition* (16) *holds, formulæ* (X) *with the amplification* (15) *define a non-field algebra in which division is always uniquely possible.*

We proceed to prove the existence of an algebra (X) for any Galois field of order p^n, $p > 2$. It suffices to take $d_5 = -1$. Then (16) becomes $c^2 - d_1^3 = 0$. We take $c = \rho^{3r}$, $d_1 = \rho^{2r}$, where r is any odd integer and ρ is a primitive root of the $GF[p^n]$. Hence c is a not-square and (16) is satisfied. In the simplified form of (19), we set

$$y = d_1^2 z, \qquad cd_3 = d_1^2(\tau + 1).$$

The factor d_1^6 may be removed. The resulting equation is

$$(20) \quad z^3 - \tau z^2 - (\tau + 3)z - 1 = 0.$$

Now d_3 and hence also τ may be chosen arbitrarily. But there exist values of τ for which (20) is irreducible in the $GF[p^n]$. Indeed, if a cubic is reducible it must have a root in the field. But (20) does not have a root 0 or -1, whatever be the value of τ. Hence, since τ enters linearly, there are at most $p^n - 2$ values of τ for which the cubic has a root in the $GF[p^n]$. Hence there are at least two irreducible cubics.*

§ 6. *Non-equivalence of the 6-tuple algebras* (II) *and* (X).

LEMMA. *Consider the field $F(\rho)$ derived from a field F by the adjunction of a root of an equation, irreducible in F,*

$$(21) \quad \rho^m = \alpha_0 + \alpha_1 \rho + \cdots + \alpha_{m-1}\rho^{m-1}.$$

The determinant $\Delta(r)$ of the algebra $F(\rho)$ has the factor

$$r_0 + r_1 \rho + \cdots + r_{m-1}\rho^{m-1}.$$

*The cubic is irreducible when $p^n = 3$, $\tau \neq 0$; $p^n = 5$, $\tau \neq 1$; $p^n = 7$, $\tau \neq 2, -2, -1$. For $p^n = 3$, there exist exactly six algebras (X); they are given by $d_5 = -1$, $d_1 = 1$, $d_3 = 0$ or 1. $d_5 = 1$, $d_1 = 0$, $d_3 = \pm 1$; $d_5 = 1$, $d_1 = \pm 1$, $d_3 = 1$.

By applying (21), we obtain relations of the form

$$\rho^{m+1} = \beta_0 + \beta_1\rho + \cdots + \beta_{m-1}\rho^{m-1}, \qquad \rho^{m+2} = \gamma_0 + \gamma_1\rho + \cdots + \gamma_{m-1}\rho^{m-1}, \cdots.$$

If $L \equiv r_0 + r_1\rho + \cdots + r_{m-1}\rho^{m-1}$ is the general element of $F(\rho)$, $\Delta(r)$ is

$$\begin{vmatrix} r_0 & r_{m-1}\alpha_0 & r_{m-2}\alpha_0 + r_{m-1}\beta_0 & r_{m-3}\alpha_0 + r_{m-2}\beta_0 + r_{m-1}\gamma_0 \cdots \\ r_1 & r_0 + r_{m-1}\alpha_1 & r_{m-2}\alpha_1 + r_{m-2}\beta_1 & r_{m-3}\alpha_1 + r_{m-2}\beta_1 + r_{m-1}\gamma_1 \cdots \\ r_2 & r_1 + r_{m-1}\alpha_2 & r_0 + r_{m-2}\alpha_2 + r_{m-1}\beta_2 & r_{m-3}\alpha_2 + r_{m-2}\beta_2 + r_{m-1}\gamma_2 \cdots \\ r_3 & r_2 + r_{m-1}\alpha_3 & r_1 + r_{m-2}\alpha_3 + r_{m-1}\beta_3 & r_0 + r_{m-3}\alpha_3 + r_{m-2}\beta_3 + r_{m-1}\gamma_3 \cdots \\ \cdot & \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{vmatrix}.$$

Multiply the 2d row by ρ, the 3d by $\rho^2, \cdots,$ the mth by ρ^{m-1}; add the products to the first row. Then the new first row is $L, \rho L, \rho^2 L, \rho^3 L, \cdots$.

THEOREM. *Let F be the $GF[p^n]$, so that $F(\rho)$ is the $GF[p^{nm}]$. When all possible sets of values in the $GF[p^n]$ are assigned to $r_0, r_1, \cdots r_{m-1}$, the determinant $\Delta(r)$ equals any chosen mark, not zero, of the $GF[p^n]$ exactly $(p^{nm}-1)/(p^n-1)$ times, and equals zero once.*

Equations (21) has the roots $\rho, \rho^{p^n}, \cdots, \rho^{p^{n(m-1)}}$. Since $\Delta(r)$ has the factor $L(\rho)$, it has the factors $L(\rho^{p^n}) = L^{p^n}, \cdots, L^{p^{n(m-1)}}$. Hence

$$\Delta(r) = L^s, \qquad s \equiv 1 + p^n + p^{2n} + \cdots + p^{n(m-1)}.$$

If a is any chosen mark, not zero, of the $GF[p^{nm}]$, the equation $L^s = a$ has exactly s roots in the $GF[p^{nm}]$, each determining one set of marks r_i of the $GF[p^n]$.

As in § 5, let F_{135} denote the determinant of a triple algebra which may be identified with the $GF[p^{3n}]$. We shall prove the

THEOREM. *If b and c are marks of the $GF[p^n]$, $p > 2$, c being a not-square and b not zero, the number of sets of solutions $r_1, \cdots, r_6$ in the $GF[p^n]$ of $F_{135}^2 - cF_{246}^2 = b$ is*

$(p^{6n}-1)/(p^n-1)$ *if* -1 *is a square;*

$(p^n+1)(p^{2n}+p^n+1)^2$ *if* -1 *and b are not-squares;*

$(p^n+1)(p^{2n}+p^n+1)(p^{2n}-3p^n+1)$ *if* -1 *is a not-square, b a square.*

There are p^n+1 sets of solutions x, y in the $GF[p^n]$ of

$$b = x^2 - cy^2 \equiv (x + Jy)^{p^n+1} \qquad (J^2 = c).$$

If -1 is a square, there are exactly two sets of solutions in which x or y vanishes, and $p^n - 1$ sets in which neither vanishes. Applying the preceding theorem for $m = 3$, we find that the number of sets of solutions $r_1, \cdots, r_6$ is

$$2(p^{2n}+p^n+1)+(p^n-1)(p^{2n}+p^n+1)^2=(p^{6n}-1)/(p^n-1).$$

If -1 is a not-square, set $-c=\gamma^2$. If b is the square of a mark β, then $x=0, y=\pm\beta/\gamma$ and $x=\pm\beta, y=0$ are solutions. Hence the number of sets r_i is

$$4(p^{2n}+p^n+1)+(p^n-3)(p^{2n}+p^n+1)^2.$$

But if b is a not-square, all p^n+1 sets of solutions have $x\neq 0, y\neq 0$, so that the number of sets r_i is $(p^n+1)(p^{2n}+p^n+1)^2$.

THEOREM. *When F is the $GF[p^n]$, $p>2$, no 6-tuple algebra* (X) *is equivalent to a 6-tuple algebra* (II).

When -1 is a not-square the statement follows from the preceding two theorems, since (§ 3) the determinant $\Delta(a)$ of (II) can be linearly transformed into that of the $GF[p^{6n}]$.

When -1 is the square of a mark η of the $GF[p^n]$, the proof will be restricted, for brevity, to an algebra (X) for which

$$c=\nu^3,\quad d_1=\nu^2,\quad d_3=\nu(2\eta-1),\quad d_5=-1 \qquad (\nu \text{ a not-square}). \tag{22}$$

Then (16) is satisfied. In (19) set $y=\nu^4 z$. Then

$$z^3-(2\eta-2)z^2-(2\eta+1)z-1=0. \tag{23}$$

This is irreducible for $p^n=5, \eta=\pm 2$; $p^n=13, \eta=5$ (but not for $\eta=-5$); $p^n=17, \eta=4$; $p^n=29, \eta=-12$ (but not for $\eta=12$). These are the earliest cases and are the only cases examined. We consider the fields for which (23) is irreducible. The only elements of the algebra (22) whose square equals ν are seen to be $\pm\rho$, where $\rho=\nu^{-2}m-\nu^{-1}\eta k$. Now $\rho(\rho i)=k+\nu^{-1}m-\nu\eta i$. Hence $\rho(\rho A)$ does not equal νA for every element A. The algebra is therefore not equivalent to (II).

The same method of pooof may be applied to other cases. Thus for $p^n=3, d_1=1, d_3=1, d_5=-1$, the only elements ρ of algebra (X) for which $\rho^2=-1$ are $\rho=\pm(i+k-m)$. But $\rho(\rho j)\neq -j$.

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CORRESPONDENCES AND THE THEORY OF GROUPS*

BY

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The object of this paper is to bring together several points connected with the general theory of correspondences and continuous groups, and to apply them to the theory of screws. Although the several results are in general not new, it seems of interest to give the accompanying presentation of the subject, as it furnishes an excellent example of the way in which the theory of continuous groups underlies the whole theory of correspondences.†

The first section is devoted to general theory. Use is made of the theorem of LIE‡ that if we have a continuous group in n variables together with an invariant equation system involving m parameters, then a 'group of the parameters' exists which is isomorphic with the given group, and it is pointed out that this theorem is fundamental in all correspondences.§ The correspondence established is that between a P_m and a P_n. Contact transformation is the particular case when $m = n$.

The screw geometry is developed from the projective group in three dimensions together with the system of equations which define a general straight line. The general theory leads at once to two important results in connection with the theory of groups:

1) *The general continuous conformal group in four dimensions is simply isomorphic with the general projective group in three.*

2) *Both these groups are simply isomorphic with the continuous projective group in five dimensions which leaves a given quadric invariant.*

There follows an immediate generalization of part of the second theorem. We have in fact the following:

3) *The general conformal group in space of n dimensions is simply isomorphic with the projective group in space of $n + 1$ dimensions which preserves a given quadric.*

These three results are due to KLEIN.‖ Some slight differences appear

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† See KLEIN, *Erlangen Programme*, 1872; Bulletin of the American Mathematical Society, vol. 2 (1893), pp. 215 sqq.

‡ See *Continuirlichen Gruppen*, pp. 549, 718.

§ KLEIN, loc. cit.

‖ See in particular *Höhere Grometrie*, vol. 1, p. 487 sqq.

because we are concerned merely with *continuous* groups; for example, our conformal group does not include inversions. The theory leads naturally to KLEIN's* correspondence between linear complexes and hyperspheres, and the properties of that correspondence are developed by *a priori* reasoning. The sphere-straight line correspondence of LIE also appears in the natural course of development. The major part of the remainder of the paper is concerned with the correspondence shown to exist between points in space of five dimensions and screws. With respect to particular results we may mention that reciprocal screws become conjugate points with respect to the fundamental quadric, and that the process of finding the resultant of any number of wrenches on given screws is shown to be equivalent to the process of finding the mass centre of given masses at given points in the five dimensional space.

The following notation is used throughout: P_m denotes a linear m dimensional manifold in ordinary space of any dimensions; S_m denotes a hypersphere of m dimensions in ordinary space of any dimensions. A 'quadric' is a locus satisfying an equation of the second degree in any space.

§ 1.

Let

$$x_i' = f_i(x_1, \cdots, x_n, a_1, \cdots, a_r) \qquad (i = 1, \cdots, n),$$

be the finite equations of a continuous group in the variables x, and let there be any equation system

$$\phi_s(x_1, \cdots, x_n, y_1, \cdots, y_m) = 0 \qquad (s = 1, \cdots, p),$$

such that, for all transformations of the group, values $y_1', \cdots, y_m'$ independent of x exist which make

$$\phi_s(x_1', \cdots, x_n', y_1', \cdots, y_m') = 0 \qquad (s = 1, \cdots, p),$$

provided the ϕ's in the unaccented variables vanish. The general theory of groups shows that the transformations for the y's form a group which is isomorphic with the original one. This group need not necessarily be of the same order as the original one. Consider for example the general linear group

$$x_i = \sum_{j=1}^{n} \alpha_{ij} x_j \qquad (i = 1, \cdots, n),$$

in conjunction with the equation system

$$x_i = y_i x_n \qquad (i = 1, \cdots, n-1),$$

* KLEIN, Mathematische Annalen, vol. 5 (1872) p. 257; see also GRACE, Transactions Cambridge Philosophical Society, vol. 16 (1898), p. 153.

The group for the x's is of order n^2, whereas the group for the y's, the general projective group in $(n-1)$ dimensions, is only of order n^2-1.

Let Xf denote any infinitesimal operator of the x group, and Yf the corresponding operator of the y group, then the structure constants are the same for the two cases, but there may be linear relations between the operators of the y group. In any case the equations $\phi = 0$ form an invariant equation system for the group whose operators are $(X+Y)f$, and this group is isomorphic with both the x and y groups.

The general condition for simple isomorphism is readily obtained. Suppose that the y group is of order $r-h$, then there must exist h infinitesimal transformations of the x group which transform the system

$$\phi_s(x_1, \cdots, x_n, y_1, \cdots, y_m) = 0 \qquad (s=1, \cdots, p),$$

into the system

$$\phi_s(x_1', \cdots, x_n', y_1, \cdots, y_m) = 0.$$

There must thus exist a subgroup of the x group for which the equations $\phi = 0$ form an invariant system, for all values of the y's. For instance, in the case considered, $x_i' = \lambda x_i$ is such a subgroup of the general linear group.

Now consider any manifold of n dimensions and let any element in it be determined by $x_1, \cdots, x_n$. Consider also any manifold of m dimensions in which an element is determined by $y_1, \cdots, y_m$. Then we say that a correspondence exists between the two manifolds such that to an element of the first corresponds a certain locus in the second and vice versâ. We can say that the x and y groups correspond, and in fact the importance of the correspondence depends largely on these groups. For example, commencing with the ordinary projective group in space of three dimensions, we may take the equation $x_3 = x_1y_1 + x_2y_2 + y_3$ as the equation system $\phi = 0$. We thus establish a correspondence between points in the space y, and planes in the space x. Now planes in the space x passing through a given line are transformed by the x group into planes passing through another line. This condition must be an invariant one for points in the y space. We see therefore that points y lying on a line transform into points lying on a line. Also the cross ratio of four x planes through a given line is invariant under all transformations of the x group, and hence the cross ratio of four y points lying on a line is invariant under all transformations of the y group. The y group is in fact also projective. This idea may be immediately applied to the general case, and we see that any relation among different x loci gives an invariant relation among the corresponding y points, and conversely. It is immediately obvious that the correspondence considered is a generalization of a contact transformation. The Lie transformation of straight

lines into spheres gives a good idea of the importance of the underlying group. We mention two theorems.*

1. There are 15 infinitesimal contact transformations of spheres into spheres, and there are 15 infinitesimal transformations of straight lines into straight lines. Each of these sets forms a group and the two groups are simply isomorphic.

2. There are 10 infinitesimal point transformations of spheres, and there are ten infinitesimal transformations of straight lines into straight lines, which leave a given linear complex invariant. The two groups are again simply isomorphic.

§ 2.

We commence with the projective group in space. This group leaves invariant the family of all straight lines. We take for the equations $\phi = 0$ the two:

$$x_1 = y_1 x_3 + y_3, \qquad x_2 = y_2 x_3 + y_4.$$

This gives a correspondence between a manifold x of three dimensions and one y of four. To a line in the x space corresponds a point in the y space, and to a point in the x space corresponds a P_2 of a particular type in the y space. Corresponding to the projective group we have a 15 parameter group in four dimensions. The first preserves intersecting lines and hence there exists an invariant relation among two specially selected y points. If two lines y and y' intersect, then

$$(y_1 - y_1')(y_4 - y_4') - (y_2 - y_2')(y_3 - y_3') = 0.$$

Hence if we regard y as fixed, all the points on a certain hypersurface of the second order must transform into points on a similar hypersurface. It is at once seen that the above relation may be made symmetrical by taking instead of the y's themselves certain linear functions of them, namely,

$$y_1 = Y_1 + iY_2, \; y_4 = Y_1 - iY_2, \; y_2 = -Y_3 - iY_4, \; y_3 = -Y_3 + iY_4.$$

We have now the relation between two points in the Y space,

$$(Y_1 - Y_1')^2 + (Y_2 - Y_2')^2 + (Y_3 - Y_3')^2 + (Y_4 - Y_4')^2 = 0,$$

invariant under the transformations of a 15 parameter group. The 15 parameter group must therefore transform all points lying on a line which meets the sphere at infinity into points on another such line. It must therefore leave this sphere invariant. It must leave invariant the relation

$$dY_1^2 + dY_2^2 + dY_3^2 + dY_4^2 = 0;$$

*See LOVETT, Annali di Mathematica, ser. III³, vol. 7 (1902), p. 39.

and therefore it must be such that

$$dY_1'^2 + dY_2'^2 + dY_3'^2 + dY_4'^2 = \rho(Y_1Y_2Y_3Y_4)(dY_1^2 + dY_2^2 + dY_3^2 + dY_4^2),$$

where Y_1', Y_2', Y_3', Y_4' is the point obtained from Y_1, Y_2, Y_3, Y_4 by any operation of the group. The transformation must therefore be conformal. Conversely, if any infinitesimal conformal transformation be performed in the Y space it will give a transformation of lines into lines in the x space and all lines through a given point will transform into lines through a point. It will therefore be a point transformation which preserves straight lines and will thus be projective. Hence we have the theorem:

The general projective group in three dimensions and the general conformal group in four are simply isomorphic.

Further, it is easily seen that a hypersurface of the second order contains an infinite number of straight lines of which ∞^1 go through any point on the surface. Conversely this property defines a hypersurface of the second order. Now a hypersphere is such a surface and the lines in question are all minimal lines. Hence a hypersphere must transform into a hypersurface of the second order under the general conformal group. But this group leaves the sphere at infinity invariant. Hence the transformed surface must be a hypersphere. We see therefore that there must exist some complex of lines in three dimensional space which is transformed into a similar complex by all projections. This complex is given by

$$Y_1^2 + Y_2^2 + Y_3^2 + Y_4^2 + 2AY_1 + 2BY_2 + 2CY_3 + 2DY_4 + E = 0,$$

where A, B, C, D, E are constants. This becomes

$$(y_1y_4 - y_2y_3) + ay_1 + by_2 + cy_3 + dy_4 + e = 0,$$

where a, b, c, d, e are constants. It is in fact a linear complex. Hence we see that projection changes a linear complex into a linear complex, and further, this is the most general point transformation that will do so. Incidentally we note that a correspondence is established between a linear complex and a hypersphere.*

Now two S_3's have one common invariant under all conformal transformations, namely, their angle of intersection. Hence there is one invariant under all projections for two linear complexes. We shall find it convenient to speak, instead of the linear complex, of the *screw* for which the lines of the linear complex are null lines. Then a screw corresponds to an S_3.

* KLEIN, Mathematische Annalen, vol. 5 (1872), p. 264.

Let there be any wrench on a given screw, and consider the moment of the system round any line. If

$$S \equiv Y_1^2 + Y_2^2 + Y_3^2 + Y_4^2 + \cdots \text{etc.} = 0$$

be the S_3 corresponding to the given line, then the moment in question is λS, where λ is defined as the "intensity" of the wrench. Now consider any two screws; associate intensity $\lambda/(\lambda+\mu)$ with the first, $\mu/(\lambda+\mu)$ with the second, and combine. We get a unit wrench on a third screw which lies on the cylindroid determined by the two original screws. We conclude that to a system of screws on a cylindroid corresponds a system of S_3's having a common sphere. Such a coaxial system includes two points S_3's. Hence there are two screws of zero pitch on a cylindroid. The two points thus determined are inverse points with respect to any one of the linear system of S_3's. Now consider any S_3 containing both these points; it corresponds to a screw which has both these lines as null lines, and is therefore reciprocal to both the screws of zero pitch; hence it is reciprocal to all the screws of the cylindroid. But any S_3 through two inverse points of S_3' cuts S_3' orthogonally. Hence if the mutual invariant of two screws vanishes the two corresponding S_3's cut at right angles.

Now consider two P_2's in the four dimensional space; if every line in one is perpendicular to every line in the other, the two P_2's are said to be at right angles. Take any S_3 passing through a fixed S_2 in a fixed P_2, and let P_3 contain this P_2. All S_3's with their centres in P_3 will have their centres on a fixed line perpendicular to the P_2 and passing through the centre of the fixed circle. Hence all S_3's through the fixed circle will have their centres in a fixed P_2 perpendicular to the P_2 of the circle, and passing through its centre (the two P_2's have of course only one point common). There will thus be a locus in the second P_2 corresponding to the point spheres of the system. Two points of this locus lie in any P_3 through the given P_2. These points lie on the line through the centre of the fixed circle perpendicular to the P_2. They are equidistant from the centre, and this distance is independent of the particular P_3 selected. Hence the locus of point spheres of the system is another circle; the two circles have the same centre, and it is easy to see that the sum of the squares of the radii is zero. The relation between the two circles is reciprocal. The whole system of S_3's is determined by any three independent ones (i. e., ones not having a common S_2). If we translate these results into three dimensions we see that given three screws there is a single infinity of lines of the zero pitch belonging to the system thus determined. There exists a single infinity of screws of zero pitch reciprocal to the whole system. To the one system corresponds a circle, and to the other another circle which is definite when the first is given. We thus see that to a regulus corresponds a circle, and to the lines meeting the rays of a regulus correspond the points of another circle. Similar considerations may

be readily applied to the system arising from the four or five screws. Four S_3's have two points common, and the reciprocal system thus includes two point S_3's; this system is in fact coaxial. The five system consists of S_3's orthogonal to a given S_3. Now suppose we fix a given screw; by an inversion we may make the corresponding S_3 a P_3, and then a correspondence is established between the lines of a given complex and points of a P_3; for any point in the P_3 corresponds to a screw of zero pitch reciprocal to the given screw, that is to say, any point in the P_3 corresponds to a line of the complex determined by the fixed screw. Any S_3 meets the P_3 in a sphere, and for a given sphere there is a single infinity of S_3's. This single infinity is determined by any given S_3 and the P_3. It follows that we may regard a sphere as corresponding to all the screws of a cylindroid which contains the P_3. Now a cylindroid is determined uniquely by two of its screws, hence the P_3 and any line determine a unique sphere. But corresponding to any sphere there are two point S_3's, that is to say two lines, and hence the correspondence between lines and spheres is a $2-1$ correspondence. The points are images in the P_3, and therefore the lines are conjugates with respect to the given complex. Now consider any two spheres. They are the intersections of the P_3 with two point S_3's; the two spheres intersect in a circle, and therefore the two point S_3's lie on a circle which is perfectly determined if the first circle is given. The rays of a regulus correspond to the circle thus determined. The regulus is not general, for it is self conjugate with respect to the fundamental screw, and the directors are therefore null lines of this screw. Suppose that the two spheres touch; both circles now become point circles, and hence each of them is a pair of minimal lines. Hence if two spheres touch, the four corresponding four dimensional points lie on two minimal lines. As the spheres are supposed general, no two points corresponding to the same sphere lie on the same minimal line, and therefore the points lie by pairs on two minimal lines. Hence the corresponding lines in the three dimensional space intersect in pairs. We note that this also includes the theorem that if two straight lines intersect their conjugates intersect. This correspondence between lines and spheres is of course that of LIE. Without inversion we should have a correspondence between lines in ordinary space and spheres in elliptic space.

We may take six S_3's as coördinates, and thus express any point by means of its powers from these S_3's. The coördinates thus used are the equivalent in four dimensions of DARBOUX's* pentaspherical coördinates in three. As a particular simplification we take the S_3's to be mutually orthogonal, and we thus see that the method of discussing screws by referring them to six co-reciprocal coördinate screws is strictly in correspondence with DARBOUX's sphere geometry.

*See DARBOUX, *Théorie des surfaces*, vol. 1, p. 213.

We take the powers mentioned to be $z_1, z_2, z_3, z_4, z_5, z_6$, and then any S_3 is given by a linear relation among the z's. The condition for orthogonality of two S_3's

$$\lambda_1 z_1 + \lambda_2 z_2 + \lambda_3 z_3 + \lambda_4 z_4 + \lambda_5 z_5 + \lambda_6 z_6 = 0$$

and

$$\lambda_1' z_1 + \lambda_2' z_2 + \lambda_3' z_3 + \lambda_4' z_4 + \lambda_5' z_5 + \lambda_6' z_6 = 0$$

is

$$\lambda_1\lambda_1' + \lambda_2\lambda_2' + \lambda_3\lambda_3' + \lambda_4\lambda_4' + \lambda_5\lambda_5' + \lambda_6\lambda_6' = 0.$$

Hence if an S_3 cuts itself orthogonally,

$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2 + \lambda_5^2 + \lambda_6^2 = 0.$$

This is therefore the condition for a point S_3. If the point S_3 cuts

$$S_3 \equiv \sum_{r=1}^{6} \lambda_r' z_r = 0$$

orthogonally,

$$\lambda_1\lambda_1' + \lambda_2\lambda_2' + \lambda_3\lambda_3' + \lambda_4\lambda_4' + \lambda_5\lambda_5' + \lambda_6\lambda_6' = 0.$$

Hence the *point* $\lambda_1, \lambda_2, \cdots, \lambda_6$ lies on S_3, and this point is therefore the centre of the point S_3. Hence among the six coördinates of any point there exists the relation

$$\sum_{r=1}^{6} z_r^2 = 0.\dagger$$

§ 3.

Consider the general relation

$$Y_1^2 + Y_2^2 + Y_3^2 + Y_4^2 + 2X_1Y_1 + 2X_2Y_2 + 2X_3Y_3 + 2X_4Y_4 + X_5 = 0$$

in conjunction with the conformal group in the space Y. This equation is invariantive, and hence it establishes a correspondence between space of four and that of five dimensions. The group for the X's changes P_4's of a particular type into P_4's of the same type. Also the Y group transforms point S_3's into point S_3's, and therefore the X group must leave invariant the manifold

$$X_1^2 + X_2^2 + X_3^2 + X_4^2 = X_5.$$

Coaxial S_3's in the Y space correspond to lines in the X space, and thus lines are transformed into lines in the X space. The X group is thus seen to be projective and to keep a particular quadric invariant. Hence the projective group in five dimensions which preserves a particular quadric is isomorphic to the general projective group in three dimensions and to the

† In connection with this S_3 geometry see DARBOUX, loc. cit.

general conformal group in four. To a general line corresponds a coaxial system of S_3's, or, since the system is determined by its common sphere, the correspondence is one between lines in five and spheres in four dimensions. To a system of lines having a common point corresponds a system of spheres lying on a common S_3. To the points of a general P_2 correspond S_3's passing through a given circle, and hence we may say that to a given P_2 in five dimensions corresponds a circle in four. If two points are conjugate with respect to the quadric, the corresponding S_3's cut orthogonally. Hence to the points of a P_4 correspond all the S_3's cutting a given S_3 orthogonally. To the points of a P_3 correspond the S_3's orthogonal to two given S_3's, or passing through two given points, and so on. Comparing the five dimensionality with the three dimensionality with which we started, we see that, e. g., to straight lines correspond points on the fundamental quadric, and inversion with respect to a linear complex is equivalent to reflection of the quadric in a given point.

It is convenient to project the fundamental quadric into an S_4. This may be done by taking new coördinates $\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6$, where

$$\rho x_1 = \xi_1,\ \rho x_2 = \xi_2,\ \rho x_3 = \xi_3,\ \rho x_4 = \xi_4,\ \rho x_5 = -(\xi_5 + 1),\ \rho = (\xi_5 - 1).$$

The quadric now becomes

$$\sum_{r=1}^{6} \xi_r^2 = 1,$$

and the S_3 in the four dimensional space becomes

$$z_6 = \xi_1 z_1 + \xi_2 z_2 + \xi_3 z_3 + \xi_4 z_4 + \xi_5 z_5,$$

where the z's are a mutually orthogonal set of S's.

The plane representation for the three system of screws due to Sir ROBERT BALL* is seen to be a particular case of this correspondence; and we note that the projective group in space which projects a three system into itself is simply isomorphic with the plane projective group that preserves a given circle.

We can at once give a geometrical interpretation in five dimensions to any screw system. To a 2-system corresponds the system of points on a line, and in general to an n system corresponds the points of a P_{n-1}. The most general projection of an n system into itself corresponds to the most general projection of a P_{n-1} into itself which preserves a given S_{n-2}. For example, the most general projection of a cylindroid into itself corresponds to the homographic transformation of a straight line which preserves two given points and so on.

We note that corresponding to a P_n $(n = 0, \cdots, 4)$, in the five dimensionality there is a reciprocal P_{4-n} with respect to the fundamental quadric. Hence an n system of screws has a $6 - n$ system of reciprocal screws.

* *Theory of Screws* (1900), chap. 15.

Now consider n screws and let there be associated with them intensities $\lambda_1, \cdots, \lambda_n$. The resultant wrench will be of intensity $\lambda_1 + \cdots + \lambda_n$, and if $\xi_{r1}, \cdots, \xi_{r5}$ are the coördinates of the point corresponding to the rth screw, the point corresponding to the resultant screw has the coördinates

$$\sum_{s=1}^{n} \xi_{st} \lambda_s \Big/ \sum_{s=1}^{n} \lambda_s \qquad (t = 1, \cdots, 5).$$

Hence finding the resultant wrench is equivalent to finding the mass centre of points ξ with associated multiples λ. The conditions for possibility of equilibrium of wrenches on n given screws follow at once from the five dimensional representation. In fact the corresponding points must obviously lie in a P_{n-1}. Hence the n screws must belong to an $n - 1$ system.

BRYN MAWR,
January, 1906.

THE TRAJECTORIES OF DYNAMICS*

BY

EDWARD KASNER

The object of this paper is to investigate the geometric character of the trajectories described by a particle moving freely in a plane under the action of any force which depends only upon the position of the particle.† The equations of motion are thus of the form ‡

$$m\frac{d^2x}{dt^2} = \phi(x, y), \qquad m\frac{d^2y}{dt^2} = \psi(x, y).$$

At the initial time, say $t = 0$, the particle may be projected from any position $x = x_0$, $y = y_0$, with any velocity, given, for instance, by the initial slope $y' = y_0'$ and the initial speed $v = v_0$. A unique trajectory is then generated. By varying the arbitrary constants x_0, y_0, y_0', v_0, we obtain in all, since each trajectory may be described from any one of its points, ∞^3 trajectories. Thus each field of force gives rise to a definite triply infinite system of curves.

The properties of such systems which are obtained in this paper are entirely general, that is, they hold for all (positional) forces. That such properties must exist is seen most readily from the fact that the differential equation of the third order which represents a dynamical system is of special form (article 1). If a triply infinite system of curves is selected arbitrarily, its differential equation will not in general be of this form, so there will exist no field of force having the given curves for trajectories. (A single trajectory has, of course, no peculiar properties.)

A first set of geometric properties is obtained by considering the ∞^1 trajectories obtained by starting particles at a given point in a given direction with all possible speeds (articles 3–11). If for each of these curves the osculating parabola is constructed at the given point, the locus of the foci of the parabolas so obtained is a circle passing through the given point. If the point is kept

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† The theory for several particles, and for motion on a surface, in space, etc., will be discussed in later papers.

‡ The functions ϕ and ψ are assumed to be uniform and to possess first and second derivatives in the region of the plane considered. The case where the force vanishes everywhere is excluded; the trajectories are then straight lines and thus constitute only a doubly infinite system.

fixed and the initial direction varied, the corresponding circle varies in a definite manner. In particular, the locus of the centers of the ∞^1 circles obtained is a conic about the given point as focus; or, what is equivalent, the envelope of the circles is itself a circle.

It is found that these properties belong to other systems of curves besides dynamical trajectories, and all such systems are explicitly determined (articles 12–16). The narrowest class of systems thus obtained involves four arbitrary functions of x, y, and is thus far more general than the dynamical class since the latter involves only two, namely, ϕ and ψ.

The next set of properties derived depends on the consideration of hyperosculating circles, i. e., circles of curvature which have contact of the third order with a trajectory (articles 17–23). The first two properties obtained hold for the more general systems referred to above. Essentially distinct properties are then obtained, first in analytic, next in geometric form.

The whole set of properties now obtained is shown to be *completely characteristic* of systems of dynamical trajectories (art. 24). The properties are sufficient as well as necessary. Any triply infinite system of curves having the properties in question can be identified with the trajectories produced by a certain (positional) force. This force is uniquely determined except for a constant factor.

Articles 9 and 10 relate to two special cases of interest. The first deals with conservative forces; here the locus, referred to above, which is in general a conic having the given point as focus, degenerates into a straight line (counted twice). The second deals with the type of force described by LECORNU as "analytic," which is of interest from the mathematical rather than from the physical point of view. The conic here becomes a circle. In both cases, the properties are characteristic (art. 25).

In the last part of the paper it is shown that collineations are the only point transformations which convert every system of trajectories into such a system. The same projective group also leaves the more general types of systems previously obtained invariant (articles 26–29). The result is finally applied to remove an essential limitation from APPELL's discussion of the transformation theory of dynamical problems.

THE DIFFERENTIAL EQUATION OF THE TRAJECTORIES.

1. Without loss of generality, we may assume that the particle is of unit mass; so that the equations defining its motion are

$$\ddot{x} = \phi(x, y), \qquad \ddot{y} = \psi(x, y), \tag{1}$$

where dots denote derivatives with respect to the time t.

To obtain the differential equation of the trajectories it is necessary to eliminate the time. Along any trajectory y is a function of x, and, if accents are used to denote differentiation with respect to x, we have the following relations between the *geometric* and *kinematic* derivatives:

$$y' = \frac{\dot{y}}{\dot{x}}, \qquad y'' = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^3}. \tag{2}$$

$$y''' = \frac{\dot{x}(\dot{x}\dddot{y} - \dot{y}\dddot{x}) - 3\ddot{x}(\dot{x}\ddot{y} - \dot{y}\ddot{x})}{\dot{x}^5}. \tag{3}$$

Two additional equations containing third derivatives are obtained by differentiating equations (1) with respect to t. These are

$$\ddot{x} = \phi_x \dot{x} + \phi_y \dot{y}, \qquad \ddot{y} = \psi_x \dot{x} + \psi_y \dot{y}, \tag{4}$$

where the subscripts denote partial derivatives.

We now have seven equations from which all the time derivatives $\dot{x}$, $\dot{y}$, $\ddot{x}$, $\ddot{y}$, $\dddot{x}$, $\dddot{y}$ may be eliminated. The result is

$$(\psi - y'\phi)y''' = \{\psi_x + (\psi_y - \phi_x)y' - \phi_y y'^2\}y'' - 3\phi y''^2. \tag{5}$$

This is the differential equation of the ∞^3 trajectories corresponding to the general field of force (1).

We introduce, for abbreviation,

$$P = \frac{\psi_x + (\psi_y - \phi_x)y' - \phi_y y'^2}{\psi - y'\phi}, \qquad Q = \frac{-3\phi}{\psi - y'\phi}, \tag{5'}$$

so that (5) may be written

$$y''' = Py'' + Qy''^2. \tag{6}$$

2. If a triply infinite system of plane curves is selected arbitrarily, its equation will not, in general, be of the special form (5) and hence no force exists of which the given curves are the trajectories. If on the other hand the equation is of form (5), it follows that there is at least one corresponding field (1). Can two fields have the same system of trajectories?

If in one field the components are ϕ, ψ and, in the other, ϕ_1, ψ_1, then, by assumption, the corresponding equations (5) must coincide; that is

$$P = P_1, \qquad Q = Q_1. \tag{7}$$

From the second equation there results

$$\frac{\psi_1}{\phi_1} = \frac{\psi}{\phi},$$

so that we may put

$$\phi_1 = \lambda\phi, \qquad \psi_1 = \lambda\psi,$$

where λ is some function of x, y.

Introducing these values in the first equation (7), we find that λ_x and λ_y both vanish, so that λ is simply a constant κ. Therefore $\phi_1 = \kappa\phi$, $\psi_1 = \kappa\psi$.

If two fields of force produce the same system of trajectories, the forces are the same except for a constant factor. The constant factor corresponds merely to a change in the unit of force. Thus the system of trajectories belonging to a given field (1) *completely defines that field.*

Osculating parabolas.

3. Just as a set of values for x, y, y', y'', that is, a differential element of second order, is pictured most simply by means of the corresponding circle of curvature, so a differential element of the third order, defined by x, y, y', y'', y''', may be pictured by the unique osculating parabola. We collect here the general formulas to be used in the subsequent discussion.

If the given element is assumed to be at the origin, so that its coördinates are $(0, 0, y', y'', y''')$, the corresponding parabola is

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0, \tag{8}$$

where

$$\begin{aligned} a &= (3y''^2 - y'y''')^2, & b &= y'''(3y''^2 - y'y'''), & c &= y'''^2, \\ d &= 9y'y''^2, & e &= -9y''^3, & f &= 0. \end{aligned} \tag{9}$$

The coördinates α, β of the focus of any parabola (8), where of course $b^2 - ac = 0$, are given by the formulas

$$\begin{aligned} 2\alpha &= \frac{1}{(be - cd)(a + c)} \{c(d^2 - e^2 + fc - fa) - 2b(de - bf)\}, \\ 2\beta &= \frac{1}{(be - cd)(a + c)} \{b(d^2 - e^2 + fc - fa) + 2c(de - bf)\}, \end{aligned} \tag{10}$$

Substituting the values (9), we have

$$\begin{aligned} 2\alpha &= \frac{-3y''\{y'''(y'^2 - 1) + 2y'(3y''^2 - y'y''')\}}{y'''^2 + (3y''^2 - y'y''')^2}, \\ 2\beta &= \frac{-3y''\{(3y''^2 - y'y''')(y'^2 - 1) - 2y'y'''\}}{y'''^2 + (3y''^2 - y'y''')^2}. \end{aligned} \tag{11}$$

4. Consider now any triply infinite system of curves, defined by a differential equation of third order

$$y''' = f(x, y, y', y'') . \tag{12}$$

Through a given point and in a given direction there pass ∞^1 curves of the system. Each of these has a definite osculating parabola at the given point. The locus of the foci of these parabolas we shall term the *focal curve*. Thus to each lineal element (x, y, y') there corresponds a definite focal curve.*

The form of the focal curve depends, of course, upon the form of the differential equation. Since x, y, y' are fixed, y''' is a certain function of y''. Substituting this in (11), α and β are expressed in terms of y''. The elimination of y'' gives then the finite equation of the required locus.

5. We proceed now to apply these considerations to systems of dynamical trajectories. The equation of such a system is of the particular form

$$y''' = Py'' + Qy''^2. \tag{6}$$

There is no difficulty in calculating the focal curves. Here P and Q have fixed values since they involve only x, y, y'. Substituting in (11) and carrying out the elimination, we find

$$2P(\alpha^2+\beta^2) + \{3(y'^2-1) - y'(y'^2+1)Q\}\alpha + \{(y'^2+1)Q - 6y'\}\beta = 0. \tag{13}$$

Here the current coördinates α, β are referred to the fixed point (x, y) as origin. Hence each focal curve is a circle passing through the given point.

THEOREM I. *If in any field of force* (1) *a particle is projected from a given point in a given direction, to each initial speed there corresponds a definite trajectory. The ∞^1 trajectories obtained by varying the initial speed are so situated that the locus of the foci of the osculating parabolas constructed at the given point is a circle passing through that point.*

Thus each field of force gives rise to a definite correspondence between the lineal elements (x, y, y') on the one hand and the circles on the other. The explicit equation of the circle corresponding to (x, y, y') is †

$$\begin{aligned} &2\{\psi_x + (\psi_y - \phi_x)y' - \phi_y y'^2\}\{(\alpha-x)^2 + (\beta-y)^2\} \\ &\quad + 3\{\psi y'^2 + 2\phi y' - \psi\}(\alpha - x) + 3\{\phi y'^2 - 2\psi y' - \phi\}(\beta - y) = 0. \end{aligned} \tag{14}$$

6. Assuming the fixed lineal element to be $x = 0$, $y = 0$, $y' = 0$, the circle (14) reduces to

$$2\psi_x(\alpha^2 + \beta^2) - 3\psi\alpha - 3\phi\beta = 0. \tag{15}$$

* The totality of focal curves for all lineal elements is, in general, a triply infinite system It may, however, in certain cases be of smaller dimensionality. To each focal system corresponds an infinitude of original systems.

† Here the α, β coördinates refer to the same axes as the x, y coördinates.

At the same time the differential equation (5) gives, for the ∞^1 trajectories defined by the given element,

$$y''' = \frac{\psi_x}{\psi} y'' - \frac{3\phi}{\psi} y''^2. \tag{16}$$

As the initial speed is changed, y'' changes, and therefore also the radius of curvature ρ, which in the present case is

$$\rho = \frac{1}{y''}.$$

The locus of centers of curvature is of course simply the normal, here the axis of y.

The focus of the osculating parabolas is simultaneously describing the circle (15), its coördinates being

$$\alpha = \frac{3y''y'''}{2(y'''^2 + 9y'^4)}, \qquad \beta = \frac{9y''^2}{2(y'''^2 + 9y'^4)}.$$

If now we let m denote the slope of the line connecting the focus with the origin, we find

$$m = \frac{3\psi y''}{\psi_x - 3\phi y''}.$$

The introduction of ρ gives

$$m = \frac{3\psi}{\psi_x \rho - 3\phi} \quad \text{or} \quad \rho = \frac{3\phi m + 3\psi}{\psi_x m}. \tag{17}$$

This is a bilinear relation between m and ρ. Therefore

*As the initial speed varies (the initial position and direction being fixed), the center of curvature of the trajectory describes a straight line (the normal) and the focus of the osculating parabola describes a circle, in such a way that the two ranges (one linear, the other circular) are homographically related.**

7. From the equation (15) we see that the circle passes through the origin in a direction whose slope is $-\psi/\phi$. But ψ/ϕ is the slope of the vector representing the force acting at the given point. These two directions are thus symmetrically situated with respect to the axis of x, which is the assumed initial direction of the trajectories.

THEOREM II. *The circle that corresponds, according to theorem I, to a given lineal element (x, y, y') is so situated that the element bisects the angle between the force acting at the given point and the tangent to the circle at that point.*

* Furthermore, the given point, which is situated on both ranges, corresponds to itself.

8. We now consider the ∞^1 circles corresponding to the ∞^1 lineal elements at a given point. We may take the given point as the origin, but the slope y' is necessarily a parameter.

Denoting by X, Y the coördinates of the center of any one of these circles, we have, from (14),

$$(18)\qquad X=-\frac{3\{\psi y'^2+2\phi y'-\psi\}}{4\{\psi_x+(\psi_y-\phi_x)y'-\phi_y y'^2\}},\qquad Y=-\frac{3\{\phi y'^2-2\psi y'-\phi\}}{4\{\psi_x+(\psi_y-\phi_x)y'-\phi_y y'^2\}}.$$

As y' varies, the point X, Y describes the curve

$$(19)\qquad \begin{aligned}(\psi_x-\phi_y)\sqrt{\phi^2+\psi^2}\sqrt{X^2+Y^2}+(\psi_x+\phi_y)(\psi X+\phi Y)\\ +(\phi_x-\psi_y)(\phi X-\psi Y)-\tfrac{3}{2}(\phi^2+\psi^2)=0.\end{aligned}$$

This is a conic with its focus at the origin. Our result is then

THEOREM III. *The locus of the centers of the ∞^1 circles associated, according to theorem I, with the ∞^1 lineal elements at a given point, is a conic with one focus at that point.*

Theorems II and III contain a complete discription of the arrangement of the ∞^1 circles considered.

It is easy to deduce from the result just obtained the form of the envelope of the circles. This is simply the director circle of the conic (19).

THEOREM III′. *The envelope of the ∞^1 circles considered is itself a circle.*

9. *Conservative force.* These results simplify considerably in two important special cases which we now consider. In the first place the conic (19) degenerates into a straight line (counted twice) if

$$(20)\qquad \phi_y-\psi_x=0.$$

This is precisely the case in which the force (1) is *conservative*; that is, when there exists a work function or potential W such that

$$(20')\qquad \phi=W_x,\qquad \psi=W_y.$$

*When the given force is conservative, the locus of the centers of the circles considered in theorem III is, for every point of the plane, a straight line. This property is characteristic of the conservative case.**

10. *Analytic force.* In the other special case the conic (19) reduces to a circle. This is so when

$$(21)\qquad \phi_x-\psi_y=0,\qquad \phi_y+\psi_x=0.$$

* When the force is not conservative (20) represents a curve, and only for the points of this curve is the corresponding conic a straight line.

These are the conditions under which ϕ, ψ are conjugate harmonic functions, i. e., under which $\phi + i\psi$ is a function of the complex variable $x + iy$. Such forces have been investigated by LECORNU,† who describes them as *analytic forces.*

When, and only when, the force is "analytic," in the sense of LECORNU, *is the conic of theorem III, for every point of the plane, a circle.*

The enveloping circle of theorem III′ then has its center at the given point. It follows that the ∞^1 focal circles corresponding to the lineal elements at any point all have the same radius.

11. In the general case there corresponds, by theorem III, a definite conic (19) to each point x, y of the plane. One focus is at the given point and the other is

$$(21)\qquad \begin{aligned} \bar{x} &= x + \frac{3\{\psi(\phi_y + \psi_x) + \phi(\phi_x - \psi_y)\}}{(\phi_y - \psi_x)^2 - \{(\phi_x - \psi_y)^2 + (\phi_y + \psi_x)^2\}}, \\ \bar{y} &= y + \frac{3\{\phi(\phi_y + \psi_x) - \psi(\phi_x - \psi_y)\}}{(\phi_y - \psi_x)^2 - \{(\phi_x - \psi_y)^2 + (\phi_y + \psi_x)^2\}}. \end{aligned}$$

Thus each field of force gives rise to a definite correspondence between the points of the plane.

In the "analytic" case every point corresponds to itself. All points $\bar{x}$, $\bar{y}$ are at infinity when

$$(\phi_x - \psi_y)^2 + (\phi_y + \psi_x)^2 - (\phi_y - \psi_x)^2 = 0,$$

that is, when,

$$(22)\qquad (\phi_x - \psi_y)^2 + 4\phi_y\psi_x = 0.$$

In this special case the conic (19) is a parabola and the envelope considered in theorem III′ is a straight line.

CONVERSE QUESTIONS.

12. We proceed now to inquire to what extent the geometric properties obtained for general dynamical systems of trajectories are characteristic of such systems. If a triply infinite system of plane curves has one or more of these properties does it follow that it corresponds to a field of force (1)? It will be shown that this is not the case. Thus certain classes of systems will be brought to light which include the dynamical class as a special case.

13. Our first problem is to convert theorem I. Find all triply infinite systems possessing the property that for every lineal element the corresponding focal curve is a circle passing through the point of the element. We shall refer to this as *property* I.

† *Sur les forces analytiques*, Journal de l'Ecole Polytecnique, vol. 55 (1885), pp. 253-274.

The only systems of curves possessing property I are those defined by a differential equation of the form

$$y''' = G(x, y, y')y'' + H(x, y, y')y''^2, \tag{23}$$

where G and H are arbitrary functions of x, y, y'.

By assumption, the foci α, β of the osculating parabolas of the ∞^1 curves defined by any lineal element x, y, y' lie on a circle,

$$A_0(\alpha^2 + \beta^2) + A_1\alpha + A_2\beta = 0,$$

through the point x, y which is taken as the origin for the α, β coördinates. Here the coefficients may involve x, y, y' in any way. Substituting the general values of α, β given in (11), we find, by direct reduction, that the differential equation is of type (23).

This type is much more general than the dynamical type (5) or (6), since in the latter P and Q are special functions of x, y, y'.

The circle corresponding to a lineal element is, for (23), given by

$$\begin{aligned} 2G(\alpha^2 + \beta^2) + \{3(y'^2 - 1) - y'(y'^2 + 1)H\}\alpha \\ + \{(y'^2 + 1)H - 6y'\}\beta = 0. \end{aligned} \tag{24}$$

It is easily shown that all systems (23) have the homographic property considered in article 6.*

14. In order to make it possible to convert theorem II, it is of course necessary to state the property there proved in a form which does not assume the existence of a field of force. We state it as

Property II. There exists for each point (x, y) of the plane a certain direction such that the angle between this direction and the tangent to the focal circle corresponding to any element (x, y, y') at the given point, is bisected by that element.

The particular direction, of course, may vary from point to point. Its slope is thus an arbitrary function ω of x, y.

The problem is to find those systems (23) which have property II. The slope of the circle (24) at the given point is

$$\frac{3(y'^2 - 1) - y'(y'^2 + 1)H}{6y' - (y'^2 + 1)H}.$$

* If no restriction is placed on the focal locus, but it is required that the line joining the given point O to the focus (α, β) shall describe a pencil homographic to the range described by the center of curvature, then the differential equation belongs to the more general type

$$y''' = \frac{F_1 + F_2 y''}{F_3 + F_4 y''} y''^2,$$

where the F's are any functions of x, y, y'. The focal locus is then found to be a quartic curve.

The bisection property is translated by the condition

$$\frac{\omega - y'}{1 + \omega y'} = \frac{y'\{6y' + (1+y'^2)H\} - 3(y'^2-1) + y'(y'^2+1)}{6y' - (y'^2+1)H + y'\{3(y'^2-1) - y'(y'^2+1)H\}}.$$

Solving this for H, we find

$$H = \frac{3}{y' - \omega}. \tag{25}$$

The only systems possessing properties I *and* II *are those of the type* (23) *for which* H *is of the special form* (25). *The differential equation is then*

$$(y' - \omega)y''' = (y' - \omega)Gy'' + 3y''^2, \tag{26}$$

where G *is any function of* x, y, y' *and* ω *is any function of* x, y.

This class is obviously more general than the dynamical class (5). In dynamical systems the fixed direction whose slope is ω is, of course, the direction of the force acting at the given point.

15. The question suggested by theorem III is that of finding systems which, in addition to property I, have

Property III. The locus of the centers of the ∞^1 circles corresponding to the elements at a given point is a conic with that point as a focus.

For the equation (23) the focal circles are given by (24). The center X, Y is then

$$X = \frac{y'(y'^2+1)H - 3(y'^2-1)}{4G}, \qquad Y = \frac{6y' - (y'^2+1)H}{4G}. \tag{27}$$

The equation of a conic with its focus at the given point, which is the origin of the X, Y coördinates, is

$$B_0\sqrt{X^2+Y^2} + B_1X + B_2Y + B_3 = 0. \tag{28}$$

The coefficients may involve x, y, y' in any way.

From (27), we have

$$\sqrt{X^2+Y^2} = \frac{(y'^2+1)\sqrt{(y'^2+1)H^2 - 6y'H + 9}}{4G}. \tag{27'}$$

Substituting in (28), we find

$$\begin{aligned} B_0(y'^2+1)\sqrt{(y'^2+1)H^2 - 6y'H + 9} + B_1\{y'(y'^2+1)H - 3(y'^2-1)\} \\ + B_2\{6y' - (y'^2+1)H\} + 4B_3G = 0. \end{aligned} \tag{28'}$$

Solving this for G, and changing the notation slightly, we express our result as follows:

The only systems possessing properties I and III are those defined by an equation of type (23) in which

$$G = \lambda_1(1+y'^2)^{\frac{1}{2}}\sqrt{(1+y'^2)H^2 - 6y'H + 9} + \lambda_2\{y'(y'^2+1)H - 3(y'^2-1)\} + \lambda_3\{6y' - (y'^2+1)H\}. \tag{29}$$

The differential equation thus involves one arbitrary function H of x, y, y' and three arbitrary functions $\lambda_1, \lambda_2, \lambda_3$ of x, y.

16. We now inquire for the systems possessing all three of the properties considered.

Putting the results obtained in the last three paragraphs together, namely, combining formulas (23), (25), and (29), and changing the notation slightly, we find that

The only systems of curves possessing properties I, II, and III are those whose differential equation is of the form

$$(y' - \omega)y''' = \{\lambda y'^2 + \mu y' + \nu\}y'' + 3y''^2, \tag{30}$$

where $\omega, \lambda, \mu, \nu$ are arbitrary functions of x, y.

The class (30) obviously includes the dynamical type (5) as a special case.* The geometric properties obtained up to this point hold for the total class (30) and hence are not by themselves sufficient to characterize dynamical systems.† We therefore proceed to obtain additional geometric properties.

Hyperosculating circles.

17. Under what conditions will the circle of curvature constructed for a trajectory at a given point have contact of the third order with the trajectory?

The differential equation of all circles is

$$(1+y'^2)y''' = 3y'y''^2. \tag{31}$$

The differential equation of the trajectories is

$$y''' = Py'' + Qy''^2. \tag{6}$$

In case of hyperosculation the values of y''' found from these equations must be the same. Hence‡

* The comparison shows that the derivatives y', y'', y''' are involved in exactly the same form. The distinction is with respect to the generality of the functions of x, y appearing as coefficients.

† It is, however, easy to show that if any ∞^2 curves passing through a point are given, such that properties I, II and III hold for that point, it is possible to find (in infinitely many ways) a field of force such that each of the ∞^2 trajectories passing through the given point has contact of the third order with one of the given curves.

‡ We disregard the factor y'' leading to straight lines; these enter as limiting cases in every system of trajectories.

$$P + Qy'' = \frac{3y'y''}{1 + y'^2}.$$

Solving for y'', and substituting from (5′), we find

$$y'' = \frac{\{\psi_x + (\psi_y - \phi_x)y' - \phi_y y'^2\}(1 + y'^2)}{3(\phi + y'\psi)}. \tag{32}$$

This equation shows that to any set of values of x, y, y' there corresponds one value of y''. Hence to each lineal element there corresponds one trajectory which is hyperosculated by its circle of curvature.

The center of curvature for any element of the second order, referred to the given point as origin, is

$$X = -\frac{y'(1 + y'^2)}{y''}, \qquad Y = \frac{1 + y'^2}{y''}. \tag{33}$$

Inversely,

$$y' = -\frac{X}{Y}, \qquad y'' = \frac{X^2 + Y^2}{Y^3}. \tag{33′}$$

The result of substituting these values in (32) is

$$\phi_y X^2 - (\phi_x - \psi_y)XY - \psi_x Y^2 + 3(\phi Y - \psi X) = 0. \tag{35}$$

This represents a conic passing through the given point.

THEOREM IV. *In each direction through a given point there passes one trajectory which has contact of the third order with its circle of curvature. The locus of the centers of the ∞^1 hyperosculating circles, obtained by varying the initial direction, is a conic passing through the given point.*

The equation of the tangent to the conic (19) at the origin is

$$\phi Y - \psi X = 0. \tag{35′}$$

The slope of this line is ψ/ϕ. Hence

THEOREM IV_1. *The conic of theorem IV passes through the given point in the direction of the force acting at that point.*

18. In the special cases considered in articles 9 and 10 we find, by means of (35), these results:

When the force (1) *is conservative, the conic associated by theorem IV with each point of the plane is an equilateral hyperbola. Conversely, if for every point the conic is an equilateral hyperbola, the force is conservative.*

The conic is a circle for each point when, and only when, the force is "analytic."

We note that the condition that (35) is a parabola leads to the same equation (22) that was obtained in article 11. Hence

The conic of theorem IV is a parabola when, and only when, this is true for the conic of theorem III.

19. We now consider the converse questions connected in the general results obtained in article 17. The property considered in theorem IV will be referred to as *property IV*. *Property* IV_1, involved in theorem IV_1, is to be stated in a form which does not assume the existence of a field of force; this may be done by replacing the direction of the force by the special direction considered in theorem II.

It is easily seen that the discussions of article 17 apply to all equations of the class (30). The locus of centers is in fact given by the equation

$$\lambda X^2 - \mu XY + \nu Y^2 + 3(Y - \omega X) = 0, \tag{36}$$

which takes the place of (35). The tangent at the origin is

$$Y - \omega X = 0. \tag{36'}$$

It follows that

Every triply infinite system of plane curves which has the properties I, II, and III also has the properties IV and IV_1.

It is thus seen that theorems IV and IV_1, relating to hyperosculating circles, are consequences of the theorems previously obtained relating to osculating parabolas. A direct derivation may be obtained by noting that a circle which has contact of third order with a trajectory must also have contact of third order with the osculating parabola of the trajectory. This is possible only when the point of contact is the vertex of the parabola. Then the radius of the circle is half the latus rectum of the parabola, and the focus is situated on the normal at the given point.

It is thus seen that the focal circle and the hyperosculating circle which correspond uniquely to any given lineal element are intimately connected. If the former is given the latter may be constructed as follows. At the given point O, erect a perpendicular to the lineal element meeting the focal circle again at a point D. Prolong this line to C so that $DC = OD$. Then the hyperosculating circle has the center C and the radius CO. If only the point O and the focal circle are given, the corresponding lineal element is obtained, according to theorem II, by bisecting the angle between the tangent to the circle and the vector representing the force acting at O. The ∞^1 focal circles corresponding to all elements at O are so situated, by theorem III, that locus of their centers is a conic with O as focus. If for each of these circles the point C is constructed as explained above, it is shown without difficulty that the locus so obtained is a conic passing through O in the direction of the force acting at O. This proves theorems IV and IV_1.

20. We now inquire for all systems which have property IV together with some of the previous properties. The discussions are omitted, only results being stated.

If a system of curves has properties I *and* IV, *its equation is of form* (23) *with* G *and* H *connected by the relation*

$$(37) \qquad (\delta_0 y'^2 + \delta_1 y' + \delta_2)\{(1+y'^2)H - 3y'\} = (\delta_3 + y'\delta_4)G,$$

where the δ*'s are arbitrary functions of* x, y.

Properties I, III, *and* IV *give rise to a class of equations, defined by the relations* (29) *and* (37), *involving eight arbitrary functions of* x, y.

Properties I, II *and* IV *define the equations*

$$(38) \qquad (y'-\omega)y''' = \frac{(\lambda y'^2 + \mu y' + \nu)(1+\omega y')}{(\gamma_1 y' + \gamma_2)} y'' + 3y''^2$$

containing five essential coefficients.

The conic considered in the fourth property is then

$$(38') \qquad \lambda X^2 - \mu XY + \nu Y^2 - 3(\gamma_1 X - \gamma_2 Y) = 0.$$

The tangent at the origin is

$$\gamma_1 X - \gamma_2 Y = 0.$$

If the slope of this is ω, (38) reduces to (30). Hence

The systems of type (30) *are completely characterized by properties* I, II, IV *and* IV_1.

Thus theorem III is a consequence of theorems I, II, IV, and IV_1. This may also be shown from the geometric considerations given in the preceding article.

Complete characterization.

21. It remains now to examine just how the dynamical systems, defined by equations of type (5), are to be distinguished in the general class (30).

If an equation of type (30) is of the special type (5), we have, by comparison,

$$(39) \qquad \omega = \frac{\psi}{\phi},$$

$$(40) \qquad \lambda = \frac{\phi_y}{\phi}, \qquad \mu = \frac{\phi_x - \psi_y}{\phi}, \qquad \nu = -\frac{\psi_x}{\phi}.$$

Substituting

$$(39') \qquad \psi = \omega\phi,$$

and introducing, for convenience,

$$(41) \qquad \bar{\phi} = \log \phi \qquad \text{or} \qquad \phi = e^{\bar{\phi}}$$

we find, from (40),

$$\lambda = \bar{\phi}_y, \qquad \mu = \bar{\phi}_x - \omega\bar{\phi}_y - \omega_y, \qquad \nu = -\omega\bar{\phi}_x - \omega_x. \tag{42}$$

From the first and third of this set,

$$\bar{\phi}_x = -\frac{\nu + \omega_x}{\omega}, \qquad \bar{\phi}_y = \lambda. \tag{43}$$

Substituting in the second of the set,

$$\lambda\omega^2 + \mu\omega + \nu + \omega_x + \omega\omega_y = 0. \tag{44}$$

It furthermore follows from (43) that

$$\lambda_x + \left(\frac{\nu + \omega_x}{\omega}\right)_y = 0. \tag{45}$$

In the dynamical case the functions λ, μ, ν, ω thus satisfy the relations (44) and (45). We now prove that these relations are sufficient.

In virtue of (45) it is possible to find a function $\bar{\phi}$ so as to satisfy both equations (43). On account of (44) the equations (42) then necessarily hold. Finally, (41) and (39′) determine a pair of functions ϕ, ψ, that is, a field of force. The integration constant connected with the determination of $\bar{\phi}$ appears merely as a constant factor in ϕ and ψ, as it should in accordance with the result obtained in article 2.

An equation of the form (30) *represents a system of dynamical trajectories when, and only when, the functions* λ, μ, ν, ω, *which appear as coefficients, satisfy the relations* (44) *and* (45).

22. The geometric interpretation of (44) is quite simple. To each point O, there corresponds, in virtue of property IV, a definite conic (36) passing through O. The slope of the tangent at O is ω. The equation of the normal is

$$X + \omega Y = 0.$$

This intersects the conic at some point N. The distance intercepted is found to be

$$ON = \frac{3(1 + \omega^2)^{\frac{3}{2}}}{\lambda\omega^2 + \mu\omega + \nu}. \tag{46}$$

Consider now the curves defined by the differential equation

$$y' = \omega(x, y). \tag{47}$$

(In the dynamical case these are the lines of force). The curve (46) which passes through O is tangent to the conic (36). Its radius of curvature,

$$\rho = \frac{(1+y'^2)^{\frac{3}{2}}}{y''},$$

is given by

$$\rho = \frac{(1+\omega^2)^{\frac{3}{2}}}{\omega_x + \omega\omega_y}. \tag{48}$$

By means of (46) and (48), the relation (44) is observed to be equivalent to

$$\frac{1}{\rho} \pm \frac{3}{ON} = 0.$$

A discussion shows that N is on the same side of O as the center of curvature. Hence the lower sign is here appropriate, so that

$$ON = 3\rho. \tag{49}$$

We thus find that all systems (30) in which the coefficients fulfill the relation (44) possess the following property, which we refer to as

Property V. The normal at any point O to the conic associated with that point according to property IV, intersects the conic again at a distance which is three times the radius of curvature of the curve (47) passing through O. The latter curve is defined geometrically by the fact that its direction at any point is the direction considered in property II, or, what is the same, the direction of the conic corresponding to that point by IV.

Every system of trajectories fulfills the relation (44) in question. Observing that ON is the radius of a certain hyperosculating circle and that (47) represents the lines of force, we may state the result as follows:

THEOREM V. *Of the trajectories which pass through a given point in the direction of the force acting at that point, there is one which has contact of the third order with its circle of curvature; the radius of curvature of this trajectory is three times the radius of curvature of the line of force passing through the given point.*

23. The interpretation of the second relation, (45), is not so simple since here the derivatives of the functions λ and ν are involved.

The intercepts of the conic (36) on the axes of coördinates are found to be

$$OA = \frac{3\omega}{\lambda}, \qquad OB = -\frac{3}{\nu}. \tag{50}$$

Hence λ and ν have the following geometric interpretations:

$$\lambda = \frac{3\omega}{OA}, \qquad \nu = -\frac{3}{OB}. \tag{50'}$$

The function ω, we have seen, represents the slope of the conic at the given point. Thus all the quantities appearing in (45) have definite geometric significance. The relation is

$$\frac{\partial}{\partial x}\left(\frac{\omega}{OA}\right)-\frac{\partial}{\partial y}\left(\frac{1}{\omega\cdot OB}\right)+\tfrac{1}{3}\frac{\partial}{\partial y}\left(\frac{\omega_x}{\omega}\right)=0. \tag{51}$$

This may be expressed in more symmetrical form as follows. At A draw a line parallel to the axis of ordinates intersecting the normal ON in some point

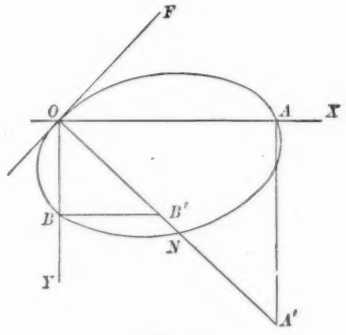

FIG. 1.

A'. Similarly, at B draw a line parallel to the axis of abscissas intersecting the normal in B'. Then

$$AA'=\frac{OA}{\omega}, \qquad BB'=-\omega\cdot OB.$$

Introducing these quantities in (51), we have the following result:

Any system (30) in which the coefficients satisfy the relation (45) has, in addition to properties I, II, IV, IV_1, the *property* VI expressed by

$$\frac{\partial}{\partial x}\frac{1}{AA'}+\frac{\partial}{\partial y}\frac{1}{BB'}+\frac{\omega\omega_{xy}-\omega_x\omega_y}{3\omega^2}=0. \tag{52}$$

In particular,

THEOREM VI. *Every system of dynamical trajectories possesses property* VI.

24. Every dynamical system has the properties I, II, III (or III′), IV, IV_1, V, VI. Conversely, since the equations (30) are characterized by either I, II, III, or I, II, IV, IV_1, it follows, from the result stated at the close of article 21, that dynamical systems may be characterized by either

$$(\Sigma_1) \qquad \text{I, II, III, V, VI,}$$

or else

(Σ_2) I, II, IV, IV_1, V, VI.

If a triply infinite system of plane curves has the set of properties (Σ_1), *or the set* (Σ_2), *it is possible to find a field of force* (1) *so that, for any initial conditions, a particle moving in the field describes one of the given curves.*

In the set (Σ_2), IV_1 may be conveniently combined with II. The modified set is

(Σ_3) IV, I, II′, V, VI,

where property II′ differs from property II by requiring the direction which is fixed at each point to be the direction of the tangent to the conic corresponding to that point according to IV.

The properties entering in each of the characteristic sets Σ_1, Σ_2, Σ_3 are all necessary in the sense that none of the properties can be derived from those which precede it in the set. The sets may then be described as *ordinally* independent. (The sets are in all probability *absolutely* independent, but the author has not attempted the requisite discussion.)

With reference to VI, it is sufficient to require this property for any one set of rectangular axes. It then necessarily holds for all rectangular axes. Hence it is really a geometric property of the system.

25. From the results obtained in articles 9 and 10, it follows that

Triply infinite systems of curves capable of generation as the trajectories of a CONSERVATIVE *force are completely characterized by the properties* (Σ_1) *together with the requirement that the conic involved in* III *shall be a straight line; or by* (Σ_3) *with the requirement that the conic of* IV *shall be an equilateral hyperbola.*

Systems which can be generated by an " ANALYTIC *" force are characterized by* (Σ_1) *with the conic of* III *required to be a circle; or by* (Σ_3) *with the same requirement imposed on the conic of* IV.

TRANSFORMATIONS

26. We consider in the following articles the effect of an arbitrary point transformation upon systems of trajectories.

Let the point transformation be

$$X = \Phi(x, y), \qquad Y = \Psi(x, y). \tag{53}$$

The first extended transformation we write in the form

$$(cy' + d)\,Y' = ay' + b, \tag{53′}$$

where

$$a = \psi_y, \qquad b = \psi_x, \qquad c = \phi_y, \qquad d = \phi_x.$$

The second extension is

$$(cy' + d)^3 Y''' = \alpha + \beta y' + \gamma y'^2 + \delta y'^3 + j y'', \tag{53''}$$

where j is the jacobian

$$j = ad - bc,$$

and

$$\alpha = \Phi_x \Psi_{xx} - \Psi_x \Phi_{xx},$$

$$\beta = \Phi_y \Psi_{xx} - \Psi_y \Phi_{xx} + 2\Phi_x \Psi_{xy} - 2\Psi_x \Phi_{xy},$$

$$\gamma = \Phi_x \Psi_{yy} - \Psi_x \Phi_{yy} + 2\Phi_y \Psi_{xy} - 2\Psi_y \Phi_{xy},$$

$$\delta = \Phi_y \Psi_{yy} - \Psi_y \Phi_{yy}.$$

Finally, the third extension is

$$\begin{aligned}(cy' + d)^5 Y'''' = {} & (cy' + d)(A_0 + A_1 y' + A_2 y'^2 + A_3 y'^3 + A_4 y'^4) \\ & - 3(\alpha + \beta y' + \gamma y'^2 + \delta y'^3)(\alpha_2 + \beta_2 y' + \gamma_2 y'^2) \\ & + (\alpha_1 + \beta_1 y' + \gamma_1 y'^2) y'' - 3cj y''^2 + j(cy' + d) y''',\end{aligned} \tag{53'''}$$

where

$$\alpha_1 = d(\beta + j_x) - 3c\alpha - 3jd_x,$$

$$\beta_1 = c(\beta + j_x) + d(2\gamma + j_y) - 3c\beta - 3j(c_x + d_y),$$

$$\gamma_1 = c(2\gamma + j_y) + 3d\delta - 3c\gamma - 3jc_y;$$

$$\alpha_2 = \Phi_{xx}, \qquad \beta_2 = 2\Phi_{xy}, \qquad \gamma_2 = \Phi_{yy};$$

$$A_0 = \alpha_x, \ A_1 = \alpha_y + \beta_x, \ A_2 = \beta_y + \gamma_x, \ A_3 = \gamma_y + \delta_x, \ A_4 = \delta_y.$$

It will be convenient to write (55'') and (53''') in more abbreviated fashion as follows:

$$Y'' = \epsilon_0 + \epsilon_1 y'', \qquad Y'''' = E_0 + E_1 y'' + E_2 y''^2 + E_3 y'''. \tag{56}$$

The coefficients here involve y' as well as x and y.

27. From (56) it is observed that an arbitrary transformation applied to an equation of type

$$y''' = G(xyy')y'' + H(xyy')y''^2 \tag{23}$$

converts it into one of the form

$$y''' = f_0(xyy') + f_1(xyy')y'' + f_2(xyy')y''^2. \tag{57}$$

For what transformations will the type (23) be invariant? In this case f_0 must vanish. This gives the condition

$$\bar{G}\epsilon_0 + \bar{H}\epsilon_0^2 = E_0, \tag{58}$$

where the bars indicate the result of applying the transformation to the functions G and H.

Since this condition is to be fulfilled for all values of G and H, we may, in particular, assume them to be arbitrary constants. Then $\bar{G}$ and $\bar{H}$ are arbitrary constants, so that ϵ_0 and E_0 must vanish. The first formula in (56) then shows that the required transformations leave invariant the differential equation $y'' = 0$. The only transformations with this property are the collineations.

Conversely, if the transformation is a collineation, the coefficients ϵ_0 and E_0 really vanish; the extended transformations are, in fact, of the form

$$\begin{aligned} (cy' + \bar{c})Y' &= ay' + b, \\ (cy' + d)^3 Y'' &= jy'', \\ (cy' + d)^5 Y''' &= (\alpha_1 + \beta_1 y' + \gamma_1 y'^2)y'' - 3cjy''^2 + j(cy' + d)y'''. \end{aligned} \tag{59}$$

Thus a collineation converts (23) into one of the same type

$$Y''' = G_1 Y'' + H_1 Y''^2,$$

the new coefficients being related to the old as follows:

$$G_1 = \frac{j(cy' + d)^2 G - (\alpha_1 + \beta_1 y' + \gamma_1 y'^2)}{j(cy' + d)^2}, \qquad H_1 = \frac{j\bar{H} + 3c(cy' + d)}{(cy' + d)^2}. \tag{60}$$

The only point transformations which convert all equations of type (23) *into equations of the same type are the projective transformations.*

Recalling the fact that a system of curves whose differential equation is of type (23) is characterized by the possession of property I, we may restate the result as follows:

Property I is invariant under collineations, but not invariant under any other point transformation.

Thus, if any number of curves which pass through a common point in a common direction are so situated that the foci of their osculating parabolas (at the given point) are located on a circle passing through the given point, the same is true after the curves are projected in any way.*

28. We now prove that *properties II and III are invariant under projective transformation.*

For a system possessing the former property (in addition, of course, to property I), the coefficient H is of the form

$$H = \frac{3}{y' - \omega(x, y)}. \tag{61}$$

* If an arbitrary point transformation is applied, the new focal locus is found to be a quartic curve of special type.

From (53′), we have

$$\overline{H} = \frac{3(cy' + d)}{ay' + b - \omega(cy' + d)}.$$

Introducing this value in the second formula of (60), we find

$$H_1 = \frac{3(a - c\omega)}{ay' + b - \omega(cy' + d)} = \frac{3}{y' - \dfrac{d\omega - b}{a - c\omega}}, \tag{61′}$$

which is of the form characteristic of property II.

In a similar fashion, it may be shown that, if G is of the form

$$G = \frac{\lambda y'^2 + \mu y' + \nu}{y' - \omega}, \tag{62}$$

the same is true of the coefficient G_1 in the transformed differential equation.

The coefficients (61) and (62) are precisely those involved in the type (30). Therefore

Any system of curves whose equation is of type (30), *that is, any system possessing properties* I, II *and* III, *is converted by projective transformation into a system of the same type. For no other transformations is this the case.*

29. We consider finally systems of dynamical trajectories, the differential equation then being of type (5).

If a point transformation is to convert every system of this kind into a system of the same kind, it is necessary that

$$\overline{P}\epsilon_0 + \overline{Q}\epsilon_0^2 - E_0 = 0. \tag{63}$$

Here $\overline{P}$, $\overline{Q}$ denote the results obtained by applying the transformation considered to the coefficients P, Q defined by (5′). This equation is to hold for all dynamical systems, i. e., for all functions ϕ and ψ.

Take first the particular force

$$\phi = 0, \qquad \psi = 1.$$

The coefficients P, Q then vanish; hence also the quantities $\overline{P}$, $\overline{Q}$. Therefore, from (63), E_0 must vanish, and the condition reduces to

$$\overline{P}\epsilon_0 + \overline{Q}\epsilon_0^2 = 0. \tag{63′}$$

Take, in the next place, the particular force

$$\phi = 1, \qquad \psi = 0.$$

Then

$$P = 0, \qquad Q = \frac{3}{y'}, \qquad \overline{P} = 0, \qquad \overline{Q} = 3\frac{cy' + d}{ay' + d}.$$

The last quantity cannot vanish. For, if it did, the function Φ, of which c and d are the partial derivatives, would reduce to a constant, so that the point transformation (53) would be degenerate. It follows therefore, from (63′), that ϵ_0 must vanish. Hence the transformation is necessarily a collineation. It is easily verified that all collineations have the required property. Hence

The only point transformations which convert every system of dynamical trajectories into such a system are the collineations of the plane. Thus the group for which the type (5) *is invariant is the eight-parameter projective group.*

Transformations involving the time.

30. The importance of projective transformations in dynamics was first indicated by Appell.* His discussion is based on the equations

$$\frac{d^2x}{dt^2} = \phi(x, y), \qquad \frac{d^2y}{dt^2} = \psi(x, y). \tag{64}$$

It is shown that for any collineation

$$x_1 = \frac{ax + by + c}{a''x + b''y + c''}, \qquad y_1 = \frac{a'x + b'y + c'}{a''x + b''y + c''}, \tag{65}$$

it is possible to find a transformation of the time, namely,

$$t_1 + \text{const.} = \frac{t}{k(a''x + b''y + c'')^2}, \tag{65′}$$

so that the equations (64), written in the new variables, are of the same form

$$\frac{d^2x_1}{dt_1^2} = \phi_1(x_1, y_1), \qquad \frac{d^2y_1}{dt_1^2} = \psi_1(x_1, y_1). \tag{66}$$

The corresponding transformation of the force components is

$$\begin{aligned} \phi_1 &= k^2(a''x + b''y + c)^2\{C'(x\psi - y\phi) + B'\phi - A'\psi\}, \\ \psi_1 &= k^2(a''x + b''y + c'')^2\{-C(x\psi - y\phi) - B\phi + A\psi\}, \end{aligned} \tag{67}$$

where the capitals denote minors in the determinant of the linear transformation (65).

The converse is discussed in a limited form: it is proved that no transformations of the type

$$x_1 = \Phi(x, y), \qquad y_1 = \Psi(x, y), \qquad t_1 + \text{const} = t\lambda(x, y), \tag{68}$$

* *De l'homographie en mécanique*, American Journal of Mathematics, vol. 12 (1890), pp. 103–114. An application to Bertrand's problem is given in vol. 13, pp. 153–158 of the same journal. Halphen had applied a particular homography in an earlier paper.

except those given by (65), (65′), can convert every system (64) into a like system.

The result holds, however, even if no special assumption is made concerning the transformation of t. We prove namely this stronger converse:

The only transformations of the type

$$x_1 = \Phi(x, y), \qquad y_1 = \Psi(x, y), \qquad t_1 = \chi(x, y, t), \tag{69}$$

for which every system (64) *is converted into a system of the same form, are the Appell transformations represented by* (65) *with* (65′).

To show this, we apply the result proved in article 29. The point transformation is necessarily a collineation, so that (69) must be of the form

$$x_1 = \frac{ax + by + c}{a''x + b''y + c''}, \qquad y_1 = \frac{ax + by + c}{a'x + b'y + c'}, \qquad t_1 = \chi(x, y, t). \tag{70}$$

Let this transformation be denoted by S' and let the Appell transformation (65), (65′) be denoted by S.

Since both transformations leave the type (64) invariant, the same is true of their combination $S'S^{-1}$, which is of the form

$$x_1 = x, \qquad y_1 = y, \qquad t_1 = f(x, y, t). \tag{71}$$

We now determine the function f. The new time derivatives are related to the old as follows:

$$\dot{x}_1 = \frac{\dot{x}}{f_x\dot{x} + f_y\dot{y} + f_t}, \qquad \ddot{x}_1 = \frac{(f_x\dot{x} + f_y\dot{y} + f_t)\ddot{x} - \dot{x}F}{(f_x\dot{x} + f_y\dot{y} + f_t)^3},$$

$$\dot{y}_1 = \frac{\dot{y}}{f_x\dot{x} + f_y\dot{y} + f_t}, \qquad \ddot{y}_1 = \frac{(f_x\dot{x} + f_y\dot{y} + f_t)\ddot{y} - \dot{y}F}{(f_x\dot{x} + f_y\dot{y} + f_t)^3},$$

where

$$F = f_x\ddot{x} + f_y\ddot{y} + f_{xx}\dot{x}^2 + 2f_{xy}\dot{x}\dot{y} + f_{yy}\dot{y}^2 + f_{xt}\dot{x} + f_{yt}\dot{y} + f_u.$$

From these formulas it is found that (71) converts (64) into

$$\begin{aligned}(f_x\dot{x} + f_y\dot{y} + f_t)\ddot{x} - \dot{x}F &= \phi(x, y)(f_x\dot{x} + f_y\dot{y} + f_t)^3,\\ (f_x\dot{x} + f_y\dot{y} + f_t)\ddot{y} - \dot{y}F &= \psi(x, y)(f_x\dot{x} + f_y\dot{y} + f_t)^3.\end{aligned} \tag{72}$$

By hypothesis these are to reduce to a system

$$\ddot{x} = \phi_1(x, y), \qquad \ddot{y} = \psi_1(x, y).$$

Substituting these values in (72), multiplying the first by $\dot{y}$, the second by $\dot{x}$, and subtracting, we find

$$(f_x\dot{x} + f_y\dot{y} + f_t)^2(\dot{y}\phi - \dot{x}\psi) = \dot{y}\phi_1 - \dot{x}\psi_1. \tag{73}$$

This is to be an identity, so that we may equate like powers of $\dot{x}, \dot{y}$. It is thus easily shown that f_x and f_y vanish, while f_t is simply a constant. Hence f is of the form

$$f = \text{const}\ t + \text{const}. \tag{74}$$

Having determined the transformation (71), we need simply combine this with S in order to obtain S'. The result is of the APPELL type, as stated in our theorem.

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ON THE AUTOMORPHIC FUNCTIONS OF THE GROUP

$(0, 3; l_1, l_2, l_3)$*

BY

RICHARD MORRIS

The chief aim of this paper is to investigate the representation of groups of the type $(0, 3; l_1, l_2, l_3)$† and of the functions belonging to them, and in particular to show that for no other groups of this type except those already studied‡ can the functions which are automorphic for the group be expressed in terms of theta functions which contain explicitly only one parameter the linear transformations of which define the group in question.

The Riemann surface, of genus p, defined by the equation

$$W^\nu = (z-a)^\alpha (z-b)^\beta (z-c)^\gamma (z-d)^\delta, \tag{1}$$

in which $\alpha+\beta+\gamma+\delta = k\nu (k = 1, 2, 3)$, α, β, γ, δ being each less than ν, depends upon only one irreducible parameter. If the branch-points vary in any way so that at the end of the variation they occupy their original positions (or are permuted in case of equal exponents), the parameter undergoes a certain transformation. Let $u_i (i = 1, 2, \cdots, p'; p' \leqq p)$ denote a set of linearly independent integrals of the first kind of genus not less than p§, and let A_i and B_i denote the results of integrating u_i around the paths a_i and b_i respectively. Then the moduli of periodicity of u_i at the cuts a_k, b_k are expressible in the form ‖ $m_k A_i + n_k B_i$, $m_k' A_i + n_k' B_i$. Let ω_i denote the quotient $A_i : B_i$. Then ω_i may be taken as the parameter of the Riemann surface. Any other such constant, ω_k, must then be expressible in terms of ω_i, either linearly by means of the bilinear relations among the moduli of the integrals of the first

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† See FRICKE and KLEIN, *Automorphe Functionen*, vol. 1, p. 353, for this notation.

‡ BURKHARDT, Mathematische Annalen, vol. 42 (1893), p. 185; HUTCHINSON, these Transactions, vol. 3 (1902), p. 1; vol. 5 (1904), p. 447; YOUNG, these Transactions, vol. 5 (1904), p. 81.

§ It is evident that p' is equal to $\phi(\nu)$ and that $\phi(\nu)$ is always even for $\nu > 2$, where $\phi(\nu)$ is the number of values that q may have, q being taken as defined in § 1. If p is greater than $\phi(\nu)$, then those integrals in excess of $\phi(\nu)$ in number, which are dependent upon values of q not relatively prime to ν, will be of genus less than p and need not be considered in this connection.

‖ See *Automorphe Functionen*, vol. 2, p. 135.

kind, or transcendentally by the vanishing of certain theta formulæ. For each monodromy of the branch-points of the Riemann surface the parameter ω_i undergoes a linear transformation, and the totality of these form a group Γ.

In the case of the elliptic Riemann surface, every function of ω, as defined above, which is automorphic for the group can be expressed by means of the elliptic theta functions with zero argument and modulus ω.

The hyperelliptic surfaces $W^3 = (z-a)(z-b)(z-c)^2(z-d)^2$ and $W^4 = (z-a)(z-b)^2(z-c)^2(z-d)^3$ were first considered from this point of view by BURKHARDT,* and were further investigated by HUTCHINSON* and YOUNG* respectively. HUTCHINSON* has also investigated the surface $W^6 = (z-a)^2(z-b)^2(z-c)^3(z-d)^5$. In these cases, the moduli for the integrals of the first kind of genus p are all linearly expressible in terms of a single parameter ω which undergoes a linear transformation for each monodromy of the branch-points. Every function of ω which is automorphic for the group of linear transformations so determined for each of the above cases is expressible in terms of hyperelliptic theta functions.

In the recently published work of FRICKE and KLEIN on automorphic functions† a section is devoted to the general consideration of automorphic functions which are associated in a similar manner with the Riemann surface (1), and the suggestion is there made that such functions would be expressible in terms of the theta functions whose moduli are those of the normal integrals of the first kind for that surface. The authors speak of the desirability of studying along this line other cases besides those investigated by BURKHARDT. Following this suggestion the present writer found, on examining a number of cases, that the moduli ω_i are not all expressible linearly in terms of one of them.

The problem then is: *What are the cases in which all the moduli of the p' integrals of the first kind of genus p are linearly expressible in terms of a single parameter?* We shall show that *there are none besides those referred to above.* The proof consists of two parts.

I. It is shown that there is a finite number of cases in which all the ω_i's reduce linearly to a single one. This is done by showing that any ω_i cannot be connected linearly with more than three others in case two of the exponents $\alpha, \beta, \gamma, \delta$ are equal, or with more than seven others in case these exponents are all different, and hence the p' parameters ω_i can reduce linearly to a single one only when p' does not exceed 4 or 8 in the respective cases.

II. It is shown that this finite number of cases reduces to those already known, by assuming that the automorphic functions are uniform, thus placing conditions upon the multipliers of the generating substitutions of the group.

*Loc. cit.

†See *Automorphe Functionen*, vol. 2, p. 131 et seq.

PART I.

§ 1. $\alpha' + \beta' + \gamma' + \delta' = \nu$ *or* 3ν.

Let q be any positive number relatively prime to ν and less than ν, and assume the relations given in the equations

$$(2)\quad q\alpha = A\nu + \alpha', \qquad q\beta = B\nu + \beta', \qquad q\gamma = C\nu + \gamma', \qquad q\delta = D\nu + \delta',$$

in which $A, B, \cdots, \alpha', \beta', \cdots$ are positive integers, and $\alpha', \beta', \cdots$ each less than ν. Then all the integrals of the first kind of genus p can be expressed in the form

$$(3)\qquad u_i = \int \frac{(z-a)^A (z-b)^B (z-c)^C (z-d)^D dz}{W^q} = \int \frac{F(z)\,dz}{W^q},$$

unless $\alpha' + \beta' + \gamma' + \delta' = \nu$. In this event, however, there is another value of q, namely $q_1 \equiv -q \pmod{\nu}$, such that the sum of the corresponding $\alpha', \beta', \gamma', \delta'$ is 3ν, giving two integrals of the first kind of the form

$$(4)\qquad u_k = \int \frac{F_1(z)\,dz}{W^{q_1}}, \qquad \text{and} \qquad u_{k+1} = \int \frac{z \cdot F_1(z)\,dz}{W^{q_1}}.$$

For, substituting $q_1 = \nu - q$ for q in equations (2), we have

$$(5)\quad \begin{aligned} (\nu-q)\alpha &= A'\nu + \alpha'', & (\nu-q)\beta &= B'\nu + \beta'', \\ (\nu-q)\gamma &= C'\nu + \gamma'', & (\nu-q)\delta &= D'\nu + \delta'', \end{aligned}$$

and by adding equations (2) and (5),

$$(6)\quad \nu\alpha = (A + A')\nu + \alpha' + \alpha'', \qquad \nu\beta = (B + B')\nu + \beta' + \beta'', \qquad \text{etc.}$$

Since $\alpha', \alpha'', \cdots$ are positive and less than ν, it follows that

$$\alpha' + \alpha'' = \nu, \qquad \beta' + \beta'' = \nu, \qquad \gamma' + \gamma'' = \nu, \qquad \delta' + \delta'' = \nu,$$

whence

$$(\alpha' + \beta' + \gamma' + \delta') + (\alpha'' + \beta'' + \gamma'' + \delta'') = 4\nu,$$

and since $\alpha' + \beta' + \gamma' + \delta' = \nu$, therefore $\alpha'' + \beta'' + \gamma'' + \delta'' = 3\nu$.

It is evident from the following that there is no linear relation between the moduli of the two integrals u_k and u_{k+1}. Let A_k, A_{k+1} and B_k, B_{k+1} denote the values of these integrals when integrated along the period paths a_1 and b_1 respectively. Since the denominators of these integrals are the same in both, the table of moduli of periodicity for u_k and u_{k+1} will have the following form:

	a_1	a_2	$\cdots$	b_1	b_2	$\cdots$
u_k	$m_1 A_k + n_1 B_k$	$m_2 A_k + n_2 B_k$	$\cdots$	$m_1' A_k + n_1' B_k$	$m_2' A_k + n_2' B_k$	$\cdots$
u_{k+1}	$m_1 A_{k+1} + n_1 B_{k+1}$	$m_2 A_{k+1} + n_2 B_{k+1}$	$\cdots$	$m_1' A_{k+1} + n_1' B_{k+1}$	$m_2' A_{k+1} + n_2' B_{k+1}$	$\cdots$

in which the coefficients m_i, m_i', n_i, n_i' are functions of the νth roots of unity, and $n_1 = -1$, $m_1 = 0$, $m_1' = 1$, $n_1' = 0$. The bilinear relation among these moduli is

$$\sum_{i=1}^{p}[(m_iA_k+n_iB_k)(m_i'A_{k+1}+n_i'B_{k+1})-(m_iA_{k+1}+n_iB_{k+1})(m_i'A_k+n_i'B_k)]=0$$

which reduces to

$$\sum_{i=1}^{p}(m_in_i'-m_i'n_i)(A_kB_{k+1}-A_{k+1}B_k)=0.$$

If the second of these factors vanishes, we get $A_k : B_k = A_{k+1} : B_{k+1}$. But this cannot be, for then two rows in the table of moduli would differ only by a factor of proportionality, which is evidently impossible. Hence

$$\sum_{i=1}^{p}(m_in_i'-m_i'n_i)=0$$

and the relation becomes identity. Accordingly ω_k is not linearly connected with ω_{k+1}. Hence

THEOREM I. *In case the sum of the residues of $q\alpha$, $q\beta$, $q\gamma$, $q\delta$ taken mod. ν is ν (q being relatively prime to ν), there always exist two integrals of the first kind u_k, u_{k+1} such that the corresponding parameters ω_k, ω_{k+1} are linearly independent.*

In what follows we shall therefore consider only those cases in which the sum of the exponents $\alpha, \beta, \gamma, \delta$ is 2ν, and also the sum of the exponents $\alpha', \beta', \gamma', \delta'$, obtained by multiplying $\alpha, \beta, \gamma, \delta$ by any integer q and reducing mod. ν is 2ν.

The different cases to be investigated are the following:

Class 1. $\alpha = \beta = \gamma = \delta$.
" 2. $\alpha = \beta = \gamma \neq \delta$.
" 3. $\alpha = \delta$, $\beta = \gamma$, $\alpha \neq \beta$, $\alpha + \beta = \nu$.
" 4. $\beta = \gamma$, $\alpha \neq \beta \neq \delta$.
" 5. $\alpha \neq \beta \neq \gamma \neq \delta$.

§ 2. *The monodromy group of the Riemann surface.*

The monodromy group for each of the classes, indicated at the end of the previous section, is next obtained.

Class 1. $\alpha = \beta = \gamma = \delta$.

Since $\alpha + \beta + \gamma + \delta = 2\nu$, $\alpha = \nu/2$. The equation (1) thus reduces to that of the elliptic Riemann surface.

Class 2. $\alpha = \beta = \gamma \neq \delta$.

A diagram for the Riemann surface in this class is given in Fig. 1, in which the branch-points are connected by branch-lines with the point at infinity.

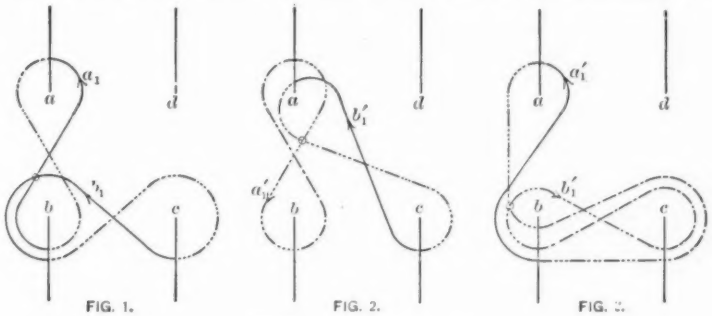


FIG. 1. FIG. 2. FIG. 3.

SHEET 1
" $1+\alpha$
" $1+2\alpha$

Let

$$u_1 = \int \frac{dz}{W}$$

be an integral of the first kind, and A_1, B_1 the values of u_1 when integrated along a_1 and b_1 respectively, and put $\sigma = e^{2\pi i/\nu}$. Also, let α_i be the value of u_1 when integrated from the point in the ith sheet corresponding to a given fixed value $z = z_0$ along a path circumscribing the branch-point a by a positive rotation and returning in the new sheet to the point corresponding to the initial value z_0 of z. Let β_i, γ_i and δ_i be similarly defined.

If

$$\alpha_1 = \frac{dz}{W},$$

then

$$\alpha_{1+a} = \int \frac{dz}{\sigma^a W} = \sigma^{-a}\alpha_1,$$

the paths of integration being as just defined, and hence,

$$A_1 = \alpha_1 - \beta_1 \qquad \text{and} \qquad B_1 = \beta_1 - \gamma_1.$$

We next determine the effect on the moduli A_1, B_1 of a monodromy of the branch-points. In the first place imagine the line joining the branch-points a and b rotated positively about its middle point until a and b are interchanged. The paths a_1 and b_1 will be deformed at the same time into a'_1 and b'_1 as indicated in Fig. 2. Denote by A'_1 and B'_1 the result of integrating u_1 along the new paths. It is readily seen that these integrals take the form

$$A'_1 = \beta_{1+a} - \alpha_{1+a} = -\sigma^{-a}A_1, \qquad B'_1 = \alpha_1 - \gamma_1 = A_1 + B_1.$$

If now $\omega_1 = A_1/B_1$, we get

$$\frac{A_1'}{B_1'} = \frac{-\sigma^{-\alpha} A_1}{A_1 + B_1},$$

whence

$$\omega_1' = \frac{-\sigma^{-\alpha}\omega_1}{\omega_1 + 1}.$$

We denote this substitution on ω_1 by V_1.

Similarly, if the line joining b and c is rotated positively about its middle point until b and c are interchanged, the paths a_1 and b_1 are deformed at the same time into a_1' and b_1' as shown in Fig. 3. The integrals along the new paths take the form

$$A_1' = \alpha_1 + \beta_{1+\alpha} - \gamma_{1+\alpha} - \beta_1 = A_1 + \sigma^{-\alpha} B_1,$$

and

$$B_1' = \gamma_{1+\alpha} - \beta_{1+\alpha} = -\sigma^{-\alpha} B_1.$$

Dividing A_1' by B_1', we get

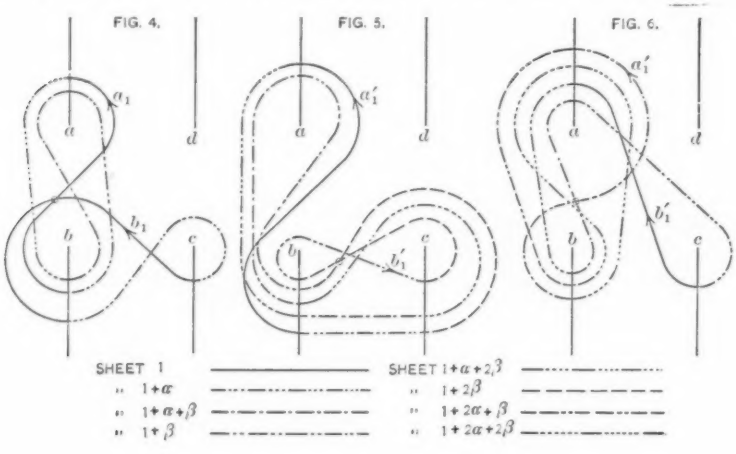
$$\omega_1' = -\sigma^{\alpha}\omega_1 - 1$$

which substitution is denoted by S_1.

Any other substitution produced by a monodromy of the branch-points may be obtained by a suitable combination of the substitutions S_1 and V_1.

Class 3. $\alpha = \delta$, $\beta = \gamma$, $\alpha \neq \beta$.

A diagram for the Riemann surface in this class is given in Fig. 4, where the branch-points, as in *Class* 2, are connected with the point at infinity.



Using the same notation as in the preceding case the values of the integral along the paths in the figure take the form

$$A_1 = \alpha_1 + \beta_{1+\alpha} - \alpha_{1+\beta} - \beta_1 \qquad \text{and} \qquad B_1 = \beta_1 - \gamma_1.$$

In determining the effect on the moduli of a monodromy of the branch-points, we first interchange b and c by a positive rotation. The paths a_1 and b_1 are transformed at the same time into those given in Fig. 5. The values of the integral along these new paths are

$$A_1' = A_1 + (\sigma^{-\beta} - \sigma^{-\alpha-\beta}) B_1 \qquad \text{and} \qquad B_1' = -\sigma^{-\beta} B_1.$$

Dividing as in the previous class, we get

$$\omega_1' = -\sigma^{\beta}\omega_1 + \sigma^{-\alpha} - 1,$$

which substitution is denoted by S_1.

In the next place rotate a about b positively. The paths a_1 and b_1 become those of Fig. 6. Integration along these new paths gives

$$A_1' = \sigma^{-(\alpha+\beta)} A_1 = A_1 \text{ (since } \alpha + \beta = \nu) \qquad \text{and} \qquad B_1' = A_1 + B_1.$$

Dividing, there is obtained

$$\omega_1' = \frac{\omega_1}{\omega_1 + 1},$$

which substitution will be denoted by V_1.

Class 4. $\beta = \gamma$, $\alpha \neq \beta \neq \delta$.

The diagram for this case is the same as that given in Fig. 4. The values of the integral along the paths in the figure are as before,

$$A_1 = \alpha_1 + \beta_{1+\alpha} - \alpha_{1+\beta} - \beta_1 \qquad \text{and} \qquad B_1 = \beta_1 - \gamma_1.$$

Interchanging b and c, the new paths are shown in Fig. 5. Integration along these paths gives,

$$A_1' = A_1 + (\sigma^{-\beta} - \sigma^{-\alpha-\beta}) B_1 \qquad \text{and} \qquad B_1' = -\sigma^{-\beta} B_1.$$

Thence, there is obtained

$$\omega_1' = -\sigma^{\beta}\omega_1 + \sigma^{-\alpha} - 1,$$

which will be denoted by S_1.

By the rotation of a about b, the transformed paths become those of Fig. 6. The new integrals are then

$$A_1' = \sigma^{-(\alpha+\beta)} A_1 \qquad \text{and} \qquad B_1' = A_1 + B_1,$$

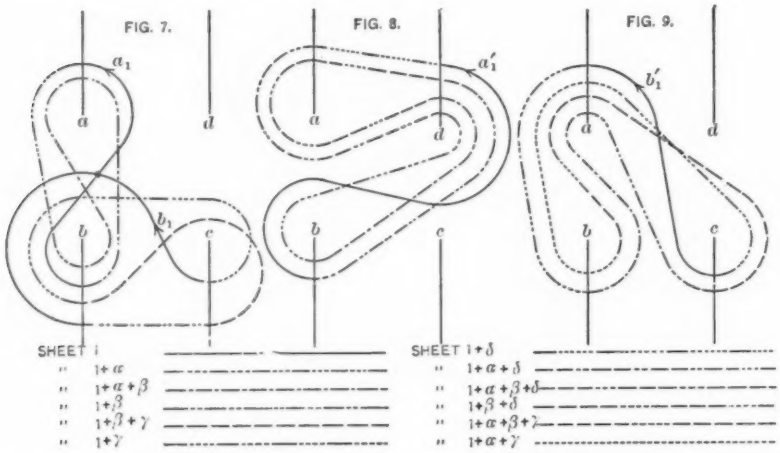
whence

$$\omega_1' = \frac{\sigma^{-(\alpha+\beta)}\omega_1}{\omega_1 + 1},$$

which we call V_1.

Class 5. $\alpha \neq \beta \neq \gamma \neq \delta$.

The diagram for this class is given in Fig. 7.



By the rotation of a about d positively, the a_1 path is transformed into that shown in Fig. 8, the b_1 path being unaltered. The integral along the a_1' path then is

$$A_1' = \sigma^{\beta+\gamma} A_1 + (\sigma^{\gamma} - \sigma^{-\alpha+\gamma}) B_1,$$

whence we get

$$\omega_1' = \sigma^{\beta+\gamma}\omega_1 + \sigma^{\gamma} - \sigma^{-\alpha+\gamma},$$

which is denoted by S_1.

Again, by the rotation of a about b, the a_1 path is transformed into the a_1' path of Fig. 6, and the b_1 path into that of Fig. 9. From this is obtained

$$A_1' = \sigma^{-(\alpha+\beta)} A_1 \qquad \text{and} \qquad B_1' = (1 - \sigma^{-\gamma}) A_1 + B_1.$$

Dividing A_1' by B_1' gives

$$\omega_1' = \frac{\sigma^{-(\alpha+\beta)}\omega_1}{(1 - \sigma^{-\gamma})\omega_1 + 1}.$$

This substitution is denoted by V_1.

§ 3. *Linear relations among the parameters* ω_i.

We next investigate under what conditions any given ω is linearly connected with other ω's and with how many others. The following is the method of procedure. When any two branch-points are interchanged, or when the line joining any two is rotated positively about its middle point until the branch-points resume their original position, a linear substitution on ω_1 is obtained, as

was seen in § 2, while simultaneous transformations are produced on all the other ω's. For each value of q there corresponds one ω, or two ω's in case $\alpha' + \beta' + \gamma' + \delta' = 3\nu$ (see § 1), and the transformation for ω_i will be obtained from that for ω_1 by replacing σ by σ^q and writing ω_i for ω_1.

If a linear relation between ω_i and ω_1 is possible, let it be assumed in the form

$$(A) \qquad \omega_1 = \frac{a\omega_i + b}{c\omega_i + d}.$$

To determine a, b, c, d the relation (A) is substituted in the transformations S_1 and V_1 for the several classes of § 2, which, after solving for ω_i', gives transformations S_i and V_i on ω_i. Next, assume that the substitutions derived by these two different processes are identical. Comparing the coefficients of ω_i, we obtain three equations from each of the transformations S_i and V_i, from which to determine the values of a, b, c, d. In general, these conditions are satisfied only by $q = \nu - 1$. But for particular values of $\alpha, \beta, \gamma, \delta$ the number of different values for q satisfying them may be at most *three for class* 4, *and at most seven for class* 5. The following is the application of the method.

Class 2. $\alpha = \beta = \gamma \neq \delta$.

The substitution S_1 or $\omega_1' = -\sigma^a\omega_1 - 1$ becomes, by replacing σ by σ^q and changing ω_1 into ω_i, S_i or $\omega_i' = -\sigma^{qa}\omega_i - 1$. Substituting (A) in S_1 and solving for ω_i', we get

$$\omega_i' = \frac{\omega_i[-ad\sigma^a - cd - bc] - bd\sigma^a - d^2 - bd}{\omega_i[ac + ac\sigma^a + c^2] + ad - bc\sigma^a + cd}.$$

Comparison of the coefficients gives the equations

$$(1) \qquad \frac{ad\sigma^a + cd + bc}{ad + bc\sigma^a + cd} = \sigma^{aq};$$

$$(2) \qquad \frac{bd\sigma^a + d^2 + bd}{ad + bc\sigma^a + cd} = 1;$$

$$(3) \qquad ac + ac\sigma^a + c^2 = 0.$$

The substitution

$$V_1 \qquad \text{or} \qquad \omega_1' = \frac{-\sigma^{-a}\omega_1}{\omega_1 + 1},$$

on changing σ into σ^q and ω_1 into ω_i, becomes

$$V_i \qquad \text{or} \qquad \omega_i' = \frac{-\sigma^{-aq}\omega_i}{\omega_i + 1}.$$

Using relation (A) and solving for ω_i', we get

$$\omega_i' = \frac{\omega_i[-ad\sigma^{-\alpha} - ab - bc] - bd\sigma^{-\alpha} - bd - b^2}{\omega_i[a^2 + ac + ac\sigma^{-\alpha}] + ab + ad + bc\sigma^{-\alpha}}.$$

Comparison of coefficients gives the equations,

$$(4) \qquad \frac{ad\sigma^{-\alpha} + ab + bc}{ab + ad + bc\sigma^{-\alpha}} = \sigma^{-\alpha q};$$

$$(5) \qquad bd\sigma^{-\alpha} + bd + b^2 = 0;$$

$$(6) \qquad \frac{a^2 + ac + ac\sigma^{-\alpha}}{ab + ad + bc\sigma^{-\alpha}} = 1.$$

There are two values of c in terms of a from equation (3), and two values of b in terms of d from equation (5), giving four different solutions of cases to be examined, which are,

Case 1, $c = -a(1+\sigma^{\alpha})$, $b = -d(1+\sigma^{-\alpha})$;
" 2, $c = -a(1+\sigma^{\alpha})$, $b = 0$;
" 3, $c = 0$, $b = -d(1+\sigma^{-\alpha})$;
" 4, $c = 0$, $b = 0$.

If each of these pairs of solutions is used in connection with equations (2) or (6) as the case may be, it is possible to express b, c, d in terms of a. The above four different solutions then become

Case 1, $c = -a(1+\sigma^{\alpha})$, $b = a(1+\sigma^{\alpha})$, $d = -a\sigma^{\alpha}$;
" 2, $c = -a(1+\sigma^{\alpha})$, $b = 0$, $d = -a\sigma^{\alpha}$;
" 3, $c = 0$, $b = a(1+\sigma^{\alpha})$, $d = -a\sigma^{\alpha}$;
" 4, $c = 0$, $b = 0$, $d = a$.

Solutions 2 and 3 are to be discarded since equation (6) is not satisfied by these values unless $\alpha = \nu/2$. From $\alpha + \beta + \gamma + \delta = 2\nu$ it then follows that δ must also equal $\nu/2$, which is impossible for this class.

If the values from solution 4 are substituted in equations (1) and (4), we get $\sigma^{\alpha q} = \sigma^{\alpha}$ and $\sigma^{-\alpha q} = \sigma^{-\alpha}$, whence is deduced the congruence $(q-1)\alpha \equiv 0 \pmod{\nu}$. Since α and ν must be relatively prime, the only value of q which satisfies this congruence is unity, and ω_i is then identical with ω_1.

Substituting the values from solution 1 in equations (1) and (4), we obtain the congruence $(q+1)\alpha \equiv 0 \pmod{\nu}$. The only value of q less than ν satisfying this is $q = \nu - 1$.

Since any one of the p' ω's may be taken for ω_1, we readily deduce the result;

THEOREM II. *For Class* 2, *any given* ω, *corresponding to a value* q_1 *of* q, *can be linearly connected with but one other* ω, *namely with that one corresponding to a value* q_2 *of* q *satisfying the relation* $q_1 + q_2 = \nu$.

If $2n = p'$ denotes the number of ω's, then there are at least n which are linearly independent.

Class 3. $\beta = \gamma$, $\alpha = \delta$, $\alpha \neq \beta$.

The transformation

$$S_1 \quad \text{or} \quad \omega_1' = -\sigma^{\beta}\omega_1 + \sigma^{-\alpha} - 1$$

becomes

$$S_i \quad \text{or} \quad \omega_i' = -\sigma^{q\beta}\omega_i + \sigma^{-q\alpha} - 1.$$

Substituting relation (A) in S_1, and solving for ω_i', as before, we find

$$\omega_i' = \frac{\omega_i[dc\sigma^{-\alpha} - dc - ad\sigma^{\beta} - bc] + d^2\sigma^{-\alpha} - d^2 - bd\sigma^{\beta} - bd}{\omega_i[ac + c^2 + ac\sigma^{\beta} - c^2\sigma^{-\alpha}] + ad + cd + bc\sigma^{\beta} - cd\sigma^{-\alpha}}.$$

Comparing coefficients, we have the equations,

(1) $$\frac{dc\sigma^{-\alpha} - dc - ad\sigma^{\beta} - bc}{ad + cd + bc\sigma^{\beta} - cd\sigma^{-\alpha}} = -\sigma^{q\beta};$$

(2) $$\frac{d^2\sigma^{-\alpha} - d^2 - bd\sigma^{\beta} - bd}{ad + cd + bc\sigma^{\beta} - cd\sigma^{-\alpha}} = -1 + \sigma^{-q\alpha};$$

(3) $$ac + c^2 + ac\sigma^{\beta} - c^2\sigma^{-\alpha} = 0.$$

The transformation

$$V_1 \text{ or } \omega_1' = \frac{\omega_1}{\omega_1 + 1}$$

becomes

$$V_i \text{ or } \omega_i' = \frac{\omega_i}{\omega_i + 1}.$$

Relation (A), applied to substitution V_1, gives

$$\omega_i' = \frac{\omega_i[ad - ab - bc] - b^2}{\omega_i a^2 + ab + ad - bc}.$$

Equations (4), (5), (6) follow from the comparison of coefficients:

(4) $$\frac{ad - ab - bc}{ad + ab - bc} = 1;$$

(5) $$-b^2 = 0;$$

(6) $$\frac{a^2}{ad + ab - bc} = 1.$$

From equation (5), $b = 0$ and from (3) we obtain $c = 0$, or $c = a(1+\sigma^{\beta})/(\sigma^{-\alpha}-1)$. By the use of $c = 0$, equation (6) gives $d = a$, and equations (1) and (2) give $\sigma^{q\beta} = \sigma^{\beta}$ and $\sigma^{q\alpha} = \sigma^{\alpha}$, whence $(q-1)\beta \equiv 0 \pmod{\nu}$ and $(q-1)\alpha \equiv 0 \pmod{\nu}$. The only value of $q < \nu$ satisfying these congruences is $q = 1$, since α and β must be relatively prime, otherwise the Riemann surface would reduce.

Taking $c = a(1+\sigma^{\beta})/(\sigma^{-\alpha}-1)$ we get, from equation (6), $d = a$, and from equations (1) and (2) $\sigma^{q\beta} = \sigma^{-\beta}$ and $\sigma^{-q\alpha} = \sigma^{\alpha}$, whence $(q+1)\beta \equiv 0 \pmod{\nu}$ and $(q+1)\alpha \equiv 0 \pmod{\nu}$. It is evident that $q = \nu - 1$ is the only value of $q < \nu$ that satisfies these last congruences, since α and β must be relatively prime.

Accordingly *the result for Class 3 is the same as for Class 2.* In particular there are at least $n = p'/2$ linearly independent ω's.

Class 4. $\beta = \gamma$, $\alpha \neq \beta \neq \delta$.

The transformation

$$S_1 \text{ or } \omega_1' = -\sigma^{\beta}\omega_1 - 1 + \sigma^{-\alpha}$$

becomes

$$S_i \quad \text{or} \quad \omega_i' = -\sigma^{q\beta}\omega_i - 1 + \sigma^{-q\alpha}.$$

Using relation (A) in S_1 and solving for ω_i', we obtain

$$\omega_i' = \frac{\omega_i[cd(\sigma^{-\alpha}-1) - ad\sigma^{\beta} - bc] + d^2(\sigma^{-\alpha}-1) - bd\sigma^{\beta} - bd}{\omega_i[ac + ac\sigma^{\beta} - c^2(\sigma^{-\alpha}-1)] + ad + bc\sigma^{\beta} - cd(\sigma^{-\alpha}-1)}.$$

Comparison as before gives the equations,

(1) $$\frac{cd(\sigma^{-\alpha}-1) - ad\sigma^{\beta} - bc}{ad + bc\sigma^{\beta} - cd(\sigma^{-\alpha}-1)} = \sigma^{q\beta};$$

(2) $$\frac{d^2(\sigma^{-\alpha}-1) - bd\sigma^{\beta} - bd}{ad + bc\sigma^{\beta} - cd(\sigma^{-\alpha}-1)} = \sigma^{-q\alpha} - 1;$$

(3) $$ac + ac\sigma^{\beta} - c^2(\sigma^{-\alpha}-1) = 0.$$

The transformation

$$V_1 \quad \text{or} \quad \omega_1' = \frac{\sigma^{-(\alpha+\beta)}\omega_1}{\omega_1 + 1}$$

becomes

$$V_i \quad \text{or} \quad \omega_i' = \frac{\sigma^{-(\alpha+\beta)q}\omega_i}{\omega_i + 1}.$$

Then using relation (A) in V_1 and solving for ω_i', we get

$$\omega_i' = \frac{\omega_i[ad\sigma^{-(\alpha+\beta)} - ab - bc] + bd[\sigma^{-(\alpha+\beta)} - 1] - b^2}{\omega_i[a^2 + ac(1-\sigma^{-\alpha-\beta})] + ad + ab - bc\sigma^{-(\alpha+\beta)}}.$$

whence we obtain the equations,

$$\frac{ad\sigma^{-(\alpha+\beta)} - ab - bc}{ad + ab - bc\sigma^{-(\alpha+\beta)}} = \sigma^{-(\alpha+\beta)q}; \tag{4}$$

$$bd\sigma^{-(\alpha+\beta)} - b^2 - bd = 0; \tag{5}$$

$$\frac{a^2 + ac\left[1 - \sigma^{-(\alpha+\beta)}\right]}{ad + ab - bc\sigma^{-(\alpha+\beta)}} = 1. \tag{6}$$

Equations (3) and (5) give two values of c in terms of a, and two values of b in terms of d, respectively, making four different solutions to be examined; these are

1, $\quad c = a\dfrac{1+\sigma^{\beta}}{\sigma^{-\alpha}-1}, \qquad b = d\left[\sigma^{-(\alpha+\beta)} - 1\right];$

2, $\quad c = 0, \qquad b = d\left[\sigma^{-(\alpha+\beta)} - 1\right];$

3, $\quad c = 0, \qquad b = 0;$

4, $\quad c = a\dfrac{1+\sigma^{\beta}}{\sigma^{-\alpha}-1}, \qquad b = 0.$

These pairs of values, each combined with equation (6), enable us to express b, c, and d in terms of a, which values of b, c, d substituted in (1), (2), and (4) give the following conditions, only two of which are independent.

For solution 1, we have

$$\sigma^{q\beta} = \sigma^{-\beta}, \qquad \sigma^{-q\alpha} = \sigma^{\alpha} \qquad \text{and} \qquad \sigma^{-(\alpha+\beta)q} = \sigma^{\alpha+\beta},$$

whence it follows that $(q+1)\alpha \equiv (q+1)\beta \equiv 0 \pmod{\nu}$.

For solution 2,

$$\sigma^{q\beta} = \sigma^{\beta}, \qquad \sigma^{-q\alpha} = \sigma^{\alpha+2\beta} \qquad \text{and} \qquad \sigma^{-(\alpha+\beta)q} = \sigma^{\alpha+\beta},$$

whence $(q-1)\beta \equiv (q+1)(\alpha+\beta) \equiv 0 \pmod{\nu}$.

For solution 3,

$$\sigma^{q\beta} = \sigma^{-\beta}, \qquad \sigma^{-q\alpha} = \sigma^{-\alpha}, \qquad \text{and} \qquad \sigma^{-(\alpha+\beta)q} = \sigma^{-(\alpha+\beta)},$$

whence $(q-1)\beta \equiv (q-1)\alpha \equiv 0 \pmod{\nu}$.

For solution 4,

$$\sigma^{q\beta} = \sigma^{-\beta}, \qquad \sigma^{-q\alpha} = \sigma^{-(\alpha+2\beta)} \qquad \text{and} \qquad \sigma^{-(\alpha+\beta)q} = \sigma^{-(\alpha+\beta)},$$

whence $(q+1)\beta \equiv (q-1)(\alpha+\beta) \equiv 0 \pmod{\nu}$.

It can be easily shown that $q = \nu - 1$ is the only value of $q < \nu$ that will satisfy the conditions of solution 1, and $q = 1$ the only value that will satisfy

the conditions of solution 3, for otherwise a factor of ν would be common to α and β, and the Riemann surface would be reducible.

Since α, β, ν cannot have a common factor, and neither can $(\alpha+\beta)$, β, ν, we have in solution 2 to consider only the following possibilities:

First, β and ν may be relatively prime;

Second, $(\alpha+\beta)$ and ν may be relatively prime;

Third, β and ν may have a common factor, and at the same time $(\alpha+\beta)$ and ν may have a common factor.

First. If β and ν are relatively prime, we find from $(q-1)\beta \equiv 0 \pmod{\nu}$ that $q=1$, which excludes this possibility, since q must be different from unity.

Second. If $(\alpha+\beta)$ and ν are relatively prime it is found from $(q+1)(\alpha+\beta)\equiv 0 \pmod{\nu}$ that $q=\nu-1$. But this will not satisfy $(q-1)\beta \equiv 0 \pmod{\nu}$ unless $\beta=\nu/2$. This value of β makes solution 2 coincide with solution 1.

Third. Suppose $\alpha+\beta=\alpha_1\alpha_2$, $\beta=\beta_1\beta_2$, $\nu=\alpha_1\beta_1\nu_1$, in which α_1 and β_1 are of necessity relatively prime. Then $(q-1)\beta=(q-1)\beta_1\beta_2=p_1\alpha_1\beta_1\nu_1$, say, whence $(q-1)\beta_2=p_1\alpha_1\nu_1$. But β_2, α_1, and ν_1 are relatively prime. Put $p_1=\beta_2 p_1'$, then $q-1=p_1'\alpha_1\nu_1$. Similarly, from $(q+1)(\alpha+\beta)=p_2\nu$ we find $q+1=p_2'\beta_1\nu_1$, if $p_2=p_2'\alpha_2$. Hence $q-1$ and $q+1$ must contain ν_1 as a factor. But, since $q-1$ and $q+1$ differ only by 2, ν_1 can have only the values 1 and 2. There is but one value of q between 1 and ν which will satisfy the equations $q-1=p_1'\alpha_1\nu_1$, and $q+1=p_2'\beta_1\nu_1$. For, suppose there are two such values, which denote by q_1 and q_2 where $q_2>q_1$. Substituting these values in the equations, we have

$$q_1-1=m_1\alpha_1\nu_1, \quad \text{and} \quad q_2-1=m_2\alpha_1\nu_1,$$
$$q_1+1=n_1\beta_1\nu_1, \quad \text{and} \quad q_2+1=n_2\beta_1\nu_1,$$

in which m_1, m_2, n_1, n_2 denote the corresponding values of p_1' and p_2'. From these equations, we derive $q_2-q_1=(m_2-m_1)\alpha_1\nu_1=(n_2-n_1)\beta_1\nu_1$, which shows that (q_2-q_1) contains α_1, β_1, and ν_1 as factors, and since these are relatively prime, (q_2-q_1) must equal $\rho\alpha_1\beta_1\nu_1$ or $\rho\nu$. Therefore $q_2=\rho\nu+q_1$, whence it follows that $q_2>\nu$.

For solution 4, the same restrictions and possibilities apply as were employed in solution 2, giving the same results.

Summarizing, and observing that the same line of argument would hold if we started with any other ω in place of ω_1, we note that in solution 1, any given ω can be linearly connected with at most one other. The corresponding values of q, say q_1 and q_2, are connected by the relation $q_1+q_2=\nu$. In solution 2, the given ω may be connected with but one other, possibly a different one from that of solution 1; in solution 3, relation (A) becomes identity; and in solution 4, there may possibly be a still different ω connected with this given one. Hence the result:

THEOREM III. *For Class* 4, *any given* ω *may be linearly connected with not more than three others.*

If it should happen for a particular Riemann surface of *class* 4 that each ω is linearly connected with three others, then the number of the $2n$ which are linearly independent would be $2n/4$ or $n/2$. Accordingly, if the number of ω's is greater than four, they are not all linearly expressible in terms of one.

Class 5. $\alpha \neq \beta \neq \gamma \neq \delta$.

The transformation

$$S_1 \quad \text{or} \quad \omega_1' = \sigma^{(\beta+\gamma)}\omega_1 + \sigma^{\gamma} - \sigma^{\gamma-\alpha}$$

becomes

$$S_i \quad \text{or} \quad \omega_i' = \sigma^{(\beta+\gamma)}\omega_i + \sigma^{q\gamma} - \sigma^{q(\gamma-\alpha)}.$$

Substituting relation (A) in S_1 and solving for ω_i' we obtain

$$\omega_i' = \frac{\omega_i\left[ad\sigma^{\beta+\gamma} + cd(\sigma^{\gamma} - \sigma^{\gamma-\alpha}) - bc\right] + bd\sigma^{\beta+\gamma} + d^2(\sigma^{\gamma} - \sigma^{\gamma-\alpha}) - bd}{\omega_i\left[ac - ac\sigma^{\beta+\gamma} - c^2(\sigma^{\gamma} - \sigma^{\gamma-\alpha})\right] + ad - bc\sigma^{\beta+\gamma} - cd(\sigma^{\gamma} - \sigma^{\gamma-\alpha})}.$$

As before, we derive the equations,

$$(1) \qquad \frac{ad\sigma^{\beta+\gamma} + cd\left[\sigma^{\gamma} - \sigma^{\gamma-\alpha}\right] - bc}{ad - cd\left[\sigma^{\gamma} - \sigma^{\gamma-\alpha}\right] - bc\sigma^{\beta+\gamma}} = \sigma^{(\beta+\gamma)q};$$

$$(2) \qquad \frac{bd(\sigma^{\beta+\gamma} - 1) + d^2\left[\sigma^{\gamma} - \sigma^{\gamma-\alpha}\right]}{ad - cd\left[\sigma^{\gamma} - \sigma^{\gamma-\alpha}\right] - bc\sigma^{\beta+\gamma}} = \sigma^{q\gamma} - \sigma^{q(\gamma-\alpha)};$$

$$(3) \qquad ac(1 - \sigma^{\beta+\gamma}) - c^2(\sigma^{\gamma} - \sigma^{\gamma-\alpha}) = 0.$$

The transformation

$$V_1 \quad \text{or} \quad \omega_1' = \frac{\sigma^{-(\alpha+\beta)}\omega_1}{(1 - \sigma^{-\gamma})\omega_1 + 1}$$

becomes

$$V_i \quad \text{or} \quad \omega_i' = \frac{\sigma^{-(\alpha+\beta)q}\omega_i}{(1 - \sigma^{-q\gamma})\omega_i + 1}.$$

Again, by substituting and solving, we have

$$\omega_i' = \frac{\omega_i\left[ad\sigma^{-(\alpha+\beta)} - bc - ab(1 - \sigma^{-\gamma})\right] + bd\sigma^{-(\alpha+\beta)} - bd - b^2(1 - \sigma^{-\gamma})}{\omega_i\left[a^2(1 - \sigma^{-\gamma}) + ac - ac\sigma^{-(\alpha+\beta)}\right] + ab(1 - \sigma^{-\gamma}) + ad - bc\sigma^{-(\alpha+\beta)}}.$$

Comparison of coefficients gives the remaining equations,

$$(4) \qquad \frac{ad\sigma^{-(\alpha+\beta)} - bc - ab(1 - \sigma^{-\gamma})}{ab(1 - \sigma^{-\gamma}) + ad - bc\sigma^{-(\alpha+\beta)}} = \sigma^{-(\alpha+\beta)q};$$

$$(5) \qquad bd\sigma^{-(\alpha+\beta)} - bd - b^2(1 - \sigma^{-\gamma}) = 0;$$

$$(6) \qquad \frac{a^2(1 - \sigma^{-\gamma}) + ac - ac\sigma^{-(\alpha+\beta)}}{ab(1 - \sigma^{-\gamma}) + ad - bc\sigma^{-(\alpha+\beta)}} = 1 - \sigma^{-q\gamma}.$$

The third and fifth equations give two values of c in terms of a and two for b in terms of d respectively, making four cases to be examined, namely,

Case 1. $c = \dfrac{a(1-\sigma^{\beta+\gamma})}{\sigma^{\gamma}-\sigma^{\gamma-\alpha}}$, $\quad b = d\dfrac{\sigma^{-(\alpha+\beta)}-1}{1-\sigma^{-\gamma}}$;

" 2. $c = 0$, $\quad b = 0$;

" 3. $c = 0$, $\quad b = d\dfrac{\sigma^{-(\alpha+\beta)}-1}{1-\sigma^{-\gamma}}$

" 4. $c = a\dfrac{1-\sigma^{\beta+\gamma}}{\sigma^{\gamma}-\sigma^{\gamma-\alpha}}$, $\quad b = 0$.

These pairs of values combined with the preceding equations give the following conditions:

Case 1. (*a*) $\sigma^{q(\beta+\gamma)} = \sigma^{-(\beta+\gamma)}$;

(*b*) $\sigma^{p(\alpha+\beta)} = \sigma^{-(\alpha+\beta)}$;

(*c*) $\sigma^{q\gamma} - \sigma^{q(\gamma-\alpha)} + \sigma^{-q\alpha} = \sigma^{\alpha} - \sigma^{-(\gamma-\alpha)} + \sigma^{-\gamma}$.

From (*a*) and (*b*), $\sigma^{q(\gamma-\alpha)} = \sigma^{-(\gamma-\alpha)}$, whence $\sigma^{q\gamma} = \sigma^{-(\gamma-\alpha)+q\alpha}$.

Substitute these in (*c*), factor, and get the equation

$$[\sigma^{(q+1)\alpha} - 1][\sigma^{-\gamma} - \sigma^{q\alpha}] = 0.$$

Case 2. (*a*) $\sigma^{q(\beta+\gamma)} = \sigma^{\beta+\gamma}$;

(*b*) $\sigma^{-(\alpha+\beta)q} = \sigma^{-(\alpha+\beta)}$;

(*c*) $\sigma^{q\gamma} + \sigma^{-q\alpha} - \sigma^{q(\gamma-\alpha)} = \sigma^{-\alpha} - \sigma^{\gamma-\alpha} + \sigma^{\gamma}$.

Equations (*a*) and (*b*) give $\sigma^{q(\gamma-\alpha)} = \sigma^{\gamma-\alpha}$, whence $\sigma^{q\gamma} = \sigma^{\gamma-\alpha+q\alpha}$.

Substituting these in (*c*) and factoring we obtain

$$[\sigma^{(q-1)\alpha} - 1][\sigma^{-q\alpha} - \sigma^{\gamma}] = 0.$$

Case 3. (*a*) $\sigma^{q(\beta+\gamma)} = \sigma^{\beta+\gamma}$;

(*b*) $\sigma^{-(\alpha+\beta)q} = \sigma^{\alpha+\beta}$.

(*c*) $\sigma^{q\gamma} - \sigma^{q(\gamma-\alpha)} + \sigma^{-q\alpha} = \sigma^{\beta} + \sigma^{\alpha+\beta+\gamma} - \sigma^{\alpha+2\beta+\gamma}$.

Then (*a*) and (*b*) give $\sigma^{q(\gamma-\alpha)} = \sigma^{\alpha+2\beta+\gamma}$, whence $\sigma^{q\gamma} = \sigma^{\alpha+2\beta+\gamma+q\alpha}$.

Substituting these in (*c*) and factoring, we obtain,

$$[\sigma^{q\alpha+\beta} - 1][\sigma^{\alpha+\beta+\gamma} - \sigma^{-q\alpha}] = 0.$$

Case 4. (*a*) $\sigma^{q(\beta+\gamma)} = \sigma^{-(\beta+\gamma)}$;

(*b*) $\sigma^{-(\alpha+\beta)q} = \sigma^{-(\alpha+\beta)}$;

(*c*) $\sigma^{q\gamma} + \sigma^{-q\alpha} - \sigma^{q(\gamma-\alpha)} = \sigma^{-\beta} + \sigma^{-(\alpha+\beta+\gamma} - \sigma^{-(\alpha+2\beta+\gamma)}$.

Using (*a*) and (*b*), $\sigma^{q(\gamma-\alpha)} = \sigma^{-(\alpha+2\beta+\gamma)}$, whence $\sigma^{q\gamma} = \sigma^{-(\alpha+2\beta+\gamma)+q\alpha}$. Finally, putting these in (*c*) and factoring, we have

$$[\sigma^{q\alpha-\beta} - 1][\sigma^{-(\alpha+\beta+\gamma)} - \sigma^{-q\alpha}] = 0.$$

For *case* 1, if $\sigma^{(q+1)\alpha} = 1$, we find from (*a*) and (*b*) that $(q+1)\alpha$, $(q+1)(\beta+\gamma)$, and $(q+1)(\alpha+\beta)$ are all divisible by ν and hence by every factor of ν. If α, $\alpha+\beta$, and $\beta+\gamma$ should all contain a factor ν_1 of ν, then β, γ, δ would also contain ν_1, and the equation of the Riemann surface would reduce. Hence $q+1$ must be divisible by all the factors of ν, that is $q+1=\nu$, or $q=\nu-1$.

Again, if $\sigma^{q\alpha-\gamma} - 1 = 0$, equations (*a*) and (*b*) give

$$q\alpha - \gamma \equiv (q+1)(\beta+\gamma) \equiv (q+1)(\alpha+\beta) \equiv 0 \pmod{\nu}.$$

For these congruences, there are only two possibilities, namely,

$$\alpha+\beta = \gamma_1\gamma_2, \qquad \beta+\gamma = \alpha_1\alpha_2, \qquad \nu = \alpha_1\gamma_1\nu_1,$$

and

$$\alpha+\beta = \gamma_1\nu_1, \qquad \beta+\gamma = \alpha_1\nu_1, \qquad \nu = \nu_1\nu_2.$$

For the first possibility, we find $q=\nu-1$, which value of q satisfies all three congruences only when $\alpha+\gamma=\nu$. For the second possibility, we can show, by a method analogous to that used in the third possibility of class 4, that there is but one value of q less than ν and different from unity that satisfies all three congruences.

There are accordingly not more than two different values of $q<\nu$ (one of which is $\nu-1$) which will satisfy the conditions of case 1.

For *case* 2, if $\sigma^{(q-1)\alpha} - 1 = 0$, we get from (*a*) and (*b*),

$$(q-1)\alpha \equiv (q-1)(\alpha+\beta) \equiv (q-1)(\beta+\gamma) \equiv 0 \pmod{\nu}.$$

The same possibilities, and only those, apply here as in the first part of *case* 1, whence $q=1$.

Again, if $\sigma^{q\alpha-\gamma} - 1 = 0$, we have the congruences

$$q\alpha + \gamma \equiv (q-1)(\alpha+\beta) \equiv (q-1)(\beta+\gamma) \equiv 0 \pmod{\nu}.$$

There are here also only two possibilities, as in the second part of *case* 1. Consideration of these shows that q can have not more than one value less than ν.

Thus there is not more than one value of q different from unity and less than ν which satisfies the conditions of *case* 2.

For *case* 3, if $\sigma^{q\alpha+\beta} - 1 = 0$, there are the congruences

$$q\alpha + \beta \equiv (q-1)(\beta+\gamma) \equiv (q+1)(\alpha+\beta) \equiv 0 \pmod{\nu},$$

the first of which may have the form $(q-1)\alpha + (\alpha+\beta) \equiv 0 \pmod{\nu}$. In these we have only three possibilities, which are:

First, $(\alpha+\beta)$ and ν relatively prime;

Second, $(\beta+\gamma)$ and ν relatively prime;

Third, $\alpha+\beta = \gamma_1\gamma_2$, $\beta+\gamma = \alpha_1\alpha_2$, $\nu = \gamma_1\alpha_1\nu_1$.

From $(q+1)(\alpha+\beta) \equiv 0 \pmod{\nu}$, q equals $\nu - 1$. This value of q satisfies $(q-1)(\beta+\gamma) \equiv 0 \pmod{\nu}$ only when $\beta+\gamma = \nu/2$, ν, or $3\nu/2$, and substituted in $q\alpha+\beta \equiv 0 \pmod{\nu}$ gives the congruence $\beta - \alpha \equiv 0 \pmod{\nu}$. But, since β and α can not differ by ν, this is impossible unless $\beta = \alpha$. This possibility is therefore to be excluded since it reduces *Class* 5 to *Class* 4.

From $(q-1)(\beta+\gamma) \equiv 0 \pmod{\nu}$, q equals 1. But this possibility must be excluded since q must be different from unity.

For the third possibility, we can easily show by methods already employed that q can not have more than one value less than ν.

Again, if $\sigma^{q\alpha+\alpha+\beta+\gamma} - 1 = 0$, we have the congruences

$$q\alpha + \alpha + \beta + \gamma \equiv (q-1)(\beta+\gamma) \equiv (q+1)(\alpha+\beta) \equiv 0 \pmod{\nu},$$

the first of which may have the different forms $(q+1)\alpha + (\beta+\gamma) \equiv q\alpha - \delta \equiv 0 \pmod{\nu}$. There are the same possibilities for this second part as for the first. From $(q+1)(\alpha+\beta) \equiv 0 \pmod{\nu}$, for the first possibility, we find $q = \nu - 1$, which imposes the condition $\beta+\gamma = \nu$. The second possibility is to be excluded because it reduces *Class* 5 to *Class* 4, since from $(q-1)(\beta+\gamma) \equiv 0$, we deduce $q = 1$, which inserted in $q\alpha - \delta \equiv 0$, gives $\alpha = \delta$. For the third possibility we can show by former methods that q can not have more than one value less than ν, which value may be different from those already obtained.

The total result in this case is that there are not more than two distinct values of $q\,(1 < q < \nu)$ possible.

Finally, for *case* 4, we have the following two sets of equations,

$$q\alpha - \beta \equiv (q+1)(\beta+\gamma) \equiv (q-1)(\alpha+\beta) \equiv 0 \pmod{\nu},$$

and

$$(q-1)\alpha - (\beta+\gamma) \equiv (q+1)(\beta+\gamma) \equiv (q-1)(\alpha+\beta) \equiv 0 \pmod{\nu}.$$

The relation $(q-1)\alpha - (\beta+\gamma) \equiv 0$ may also be written $q\alpha + \delta \equiv 0$. There are the same possibilities for each of these two sets of equations as in the previous case, for which there are similar conclusions, namely, in case neither $(\beta+\gamma)$ nor $(\alpha+\beta)$ are relatively prime to ν there are not more than two values of $q\,(1 < q < \nu)$ which satisfy all the conditions.

Summarizing, we find that *in each of the cases 1, 3, 4 there may be two distinct values of q different from unity and less than ν, and only one value of q in case 2*, one of these seven values of q being $\nu - 1$. Hence,

THEOREM IV. *For Class 5, any given ω cannot be linearly connected with more than seven others.*

If it should happen that all the ω's under consideration should separate into sets of 8, those in each set being linearly connected, then the number of those linearly independent would be $2n/8$ or $n/4$. Accordingly, if the number of ω's is greater than eight, they are not all linearly expressible in terms of one.

As a result of the investigation of these five classes, we conclude *that only a finite number of Riemann surfaces can exist for which all the ω's are linearly connected.*

PART II.

In order still further to reduce the finite number of cases which exist, we assume that the automorphic functions under consideration shall be uniform, which requires that the fundamental region for the group of substitutions of any ω_i must be a polygon, the angles at whose vertices are aliquot parts of 2π. This imposes certain conditions on the generating substitutions of the group. Suppose such a transformation to be written in the form

$$\frac{\omega' - f_1}{\omega' - f_2} = K \frac{\omega - f_1}{\omega - f_2}$$

where f_1 and f_2 are the two fixed points. Then for the generating substitutions of the groups under consideration, the "multiplier," K, must have the form

$$e^{2\pi i(p \pm 1/l)},$$

in which p and $l (l > 1)$ are integers. The method of procedure will be made clear by the consideration of *Class* 4, $\beta = \gamma$.

§ 1. *Values of l_1, l_2, and l_3 of Class 4.*

If the transformations of this class are compounded, there is obtained the transformation $T_1 = S_1 V_1$, namely,

$$\omega_1' = \frac{-\omega_1 + \sigma^{\alpha} - 1}{\omega_1 + 1}.$$

The value l_1 of l for this substitution is therefore 2.

Since the multiplier for the substitution

$$S_1 \qquad \text{or} \qquad \omega_1' = -\sigma^{\beta}\omega_1 + (\sigma^{-\alpha} - 1),$$

is $-\sigma^{\beta} = -e^{2\pi i\beta/\nu}$, it follows from the above assumption in regard to K, that β must satisfy the relation

$$\frac{\beta}{\nu} + \tfrac{1}{2} = p \pm \frac{1}{l_2}, \tag{1}$$

where l_2 denotes the values of l for S_1. From the inequality $0 < \beta < \nu$ follow

$$-\tfrac{1}{2} < \frac{2\beta - \nu}{2\nu} < \tfrac{1}{2} \qquad \text{and} \qquad \tfrac{1}{2} < \frac{2\beta + \nu}{2\nu} < \tfrac{3}{2}.$$

Hence, since $1/l_2$ is positive and does not exceed $\frac{1}{2}$, formula (1) reduces to

$$\frac{2\beta - \nu}{2\nu} = \pm \frac{1}{l_2}. \tag{2}$$

It is evident that l_2 cannot equal 2, as otherwise β would be divisible by ν. We proceed, then, to investigate the case $l_2 > 2$. The substitution

$$S_i \qquad \text{or} \qquad \omega_i' = -\sigma^{\beta'}\omega_i + (\sigma^{-\alpha'} - 1),$$

which is simultaneous with S_1, must also satisfy the same conditions, and hence

$$\frac{2\beta' - \nu}{2\nu} = \pm \frac{1}{l_2},$$

from which follows $\beta' = k_2(l_2 \pm 2)$. There are two cases to be considered, according as l_2 is *odd* or *even*.

We first take l_2 *odd*. The fraction $(2\beta - \nu)/2\nu$ will reduce to the form $\pm 1/l_2$ only when ν is even. We may therefore assume

$$\nu = 2k_2 l_2 \qquad \text{and} \qquad \beta = k_2(l_2 \pm 2). \tag{3}$$

From (2), part I, § 1, $q\beta$ takes the form $m\nu + \beta'$, whence, from the preceding relations,

$$\beta' = qk_2(l_2 \pm 2) - 2mk_2 l_2.$$

Comparing this equation with $\beta' = k_2(l_2 \pm 2)$, we have

$$qk_2(l_2 \pm 2) - 2mk_2 l_2 = k_2(l_2 \pm 2).$$

whence

$$\pm q \pm 1 = l_2 m + (1 - q)l_2/2.$$

Hence $q \equiv \pm 1 \pmod{l_2}$.

It is evident that l_2 may equal 3, since all values of q not divisible by 3 satisfy the congruence $q \equiv \pm 1 \pmod 3$.

On account of theorem III, the number of parameters ω cannot exceed 4, that is, the number of values of q, greater than unity and less than ν which are relatively prime to ν, must not exceed 3. Hence ν cannot exceed 12, and

since l_2 is an odd factor of ν, it cannot exceed 3, except that l_2 may be infinite in case $\beta = \nu/2$.

Next, take l_2 even and equal to $2\lambda_2$. From (2) p. 444, we may assume

$$\nu = k_2\lambda_2 \qquad \text{and} \qquad \beta = \frac{k_2(\lambda_2 \pm 1)}{2}. \tag{4}$$

Accordingly

$$q\beta = qk_2\frac{(\lambda_2 \pm 1)}{2} = m\nu + \beta',$$

whence

$$\beta' = qk_2\frac{(\lambda_2 \pm 1)}{2} - mk_2\lambda_2.$$

Since β and β' are required to satisfy the same condition (2), we have

$$\frac{k_2(\lambda_2 \pm 1)}{2} = qk_2\frac{(\lambda_2 \pm 1)}{2} - mk_2\lambda_2,$$

whence $\pm q \pm 1 = \lambda_2(1 + 2m - q)$, or $q \equiv \pm 1 \pmod{\lambda_2}$. This congruence is satisfied by all allowable values of q when λ_2 equals 2 or 3, that is, when $l_2 = 4$ or 6. The only other possible values for λ_2 are 4 and 6, and for ν, 4, 6, 8 and 12. The cases $\nu = 4$ and 6 give nothing new. For $\nu = 8$, $\lambda_2 = 4$, the condition $2\beta = k_2(\lambda_2 \pm 1)$ gives $\beta = 3$ or 5. Since β' satisfies the same relation, it can take no additional values. But for $\beta = 3$, $q = 3$, we find $\beta' = 1$. If $\nu = 12$, $\lambda_2 = 4$, then $2\beta = 3(4 \pm 1)$ does not give an integer value for β. Thus it is seen that $\lambda_2 = 4$ is impossible. In a similar manner, it may be shown that $\lambda_2 = 6$ is also impossible. Hence the only possible values for l_2 are 3, 4, 6.

Since the substitution V_1 has the fixed points $(\sigma^{-\alpha-\beta} - 1)$ and 0, it may be written in the following form:

$$\frac{\omega_1' - \sigma^{-\alpha-\beta} + 1}{\omega_1'} = K\frac{\omega_1 - \sigma^{-\alpha-\beta} + 1}{\omega_1}.$$

from which we deduce

$$K = \sigma^{(\alpha+\beta)} = e^{2\pi i(\alpha+\beta)/\nu}.$$

But, as stated for the substitution S_1, this must be of the form $e^{2\pi i(p \pm 1/l_3)}$, whence

$$\frac{\alpha + \beta}{\nu} = \pm\frac{1}{l_3} + p. \tag{5}$$

Now from the inequalities $0 < \alpha < \nu$, and $0 < \beta < \nu$, we get $0 < (\alpha + \beta)/\nu < 2$. Hence four values are possible for $\alpha + \beta$, namely,

$$\frac{\alpha + \beta}{\nu} = \pm\frac{1}{l_3} + 1, \qquad \frac{\alpha + \beta}{\nu} = -\frac{1}{l_3} + 2, \qquad \frac{\alpha + \beta}{\nu} = \frac{1}{l_3}.$$

The fraction $1/l_3$ may be put in the more general form k_3/l_3k_3, whence we find $\nu = k_3l_3$. Then the four possible values for $\alpha + \beta$ are given in the equations

$$(6) \qquad \begin{cases} (a), & \alpha + \beta = k_3(l_3 \pm 1), \\ (b), & \alpha + \beta = k_3(2l_3 - 1), \\ (c), & \alpha + \beta = k_3. \end{cases}$$

Since $q\alpha = m\nu + \alpha'$, and $q\beta = n\nu + \beta'$, where m and n are positive integers or zero, we have

$$(a') \quad \alpha' + \beta' = qk_3(l_3 \pm 1) - (m + n)\nu,$$

$$(b') \quad \alpha' + \beta' = qk_3(2l_3 - 1) - (m + n)\nu,$$

$$(c') \quad \alpha' + \beta' = qk_3 - (m + n)\nu.$$

And as $\alpha + \beta$ and $\alpha' + \beta'$ must have the same form, four possibilities are to be considered.

First, suppose $\alpha + \beta = k_3$, then $\alpha' + \beta' = qk_3 - (m + n)\nu$. But $\alpha' + \beta'$ must be one of the forms (a), (b), (c) of (6). Taking the condition (a), we get $qk_3 - (m + n)\nu = k_3(l_3 \pm 1)$, which by the use of $\nu = k_3l_3$, becomes $k_3(q \pm 1) = k_3l_3(m + n + 1)$, whence $q \equiv \pm 1 \pmod{l_3}$. Taking the condition (b), we have $qk_3 - (m + n)\nu = k_3(2l_3 - 1)$, which gives $q \equiv -1 \pmod{l_3}$. Taking condition (c), there results $qk_3 - (m+n)\nu = k_3$, from which $q \equiv 1 \pmod{l_3}$.

Proceeding in like manner with the values of $\alpha + \beta$ given in (b) and (c) of (6) we obtain the same conditions for q as before. Therefore, all the numbers q must satisfy the congruence $q \equiv \pm 1 \pmod{l_3}$.

It is evident that l_3 may equal 2, 3, 4, 6, since all the numbers q not containing 2, 3, 4, or 6 as a factor will satisfy the above congruence. If $l_3 = 5$, we obtain $\nu = 5$, 10, 15, etc. But for these values of ν some of the numbers q do not satisfy $q \equiv \pm 1 \pmod{5}$. It is not necessary to examine for values of ν greater than 10, since the number of values of q will then exceed 4 (see theorem III). Hence l_3 cannot equal 5. In the same way it can be shown that l_3 cannot equal any number greater than 6.

The values $l_1 = 2$, $l_2 = 3$, 4, 6, and $l_3 = 2$, 3, 4, 6, in those combinations only which satisfy the inequality

$$(7) \qquad \frac{1}{l_1} + \frac{1}{l_2} + \frac{1}{l_3} < 1$$

give us three sets, namely, $l_1 = 2$, $l_2 = 4$, $l_3 = 6$; $l_1 = 2$, $l_2 = 6$, $l_3 = 4$; $l_1 = 2$, $l_2 = 6$, $l_3 = 6$. Using the first set in connection with equations (4) and (6) of this section, and $\nu = k_3l_3$, we obtain the Riemann surface $W^{12} = (z - a)(z - b)^9(z - c)^9(z - d)^5$. For $\nu = 12$, q has the values 1, 5, 7,

11, which are found to satisfy the congruence conditions, of § 3, class 4, part I. Thus the *necessary* conditions that the four ω's may be linearly dependent are fulfilled. These, however, are not *sufficient*, as it is seen by actually working out the table of moduli that two of the ω's are linearly independent. Hence this surface is to be excluded. In a similar manner using the second set of values for the l's, we obtain the Riemann surface $W^{12}=(z-a)(z-b)^8(z-c)^8(z-d)^7$, which must also be excluded, for the same reason as before. The third set of values for the l's gives the surface $W^6=(z-a)^3(z-b)^2(z-c)^2(z-d)^5$, which has already been treated (see footnote p. 425).

If l_2 equals infinity, we get the three sets $l_1=2$, $l_2=\infty$, $l_3=3$; $l_1=2$, $l_2=\infty$, $l_3=4$; $l_1=2$, $l_2=\infty$, $l_3=6$, which however give nothing new. Hence we conclude *that no new cases of the kind sought for are found in Class* 4.

§ 2. *Values of* l_1, l_2, l_3 *for class* 5.

By compounding the transformations of this class there is obtained the transformation $T_1=V_1^{-1}S_1^{-1}$, namely,

$$\omega_1'=\frac{\omega_1+\sigma^{-\alpha+\gamma}-\sigma^{\gamma}}{(\sigma^{-\gamma}-1)\omega_1+\sigma^{-\alpha}+\sigma^{\gamma}-1}.$$

By the method employed for substitution V_1 in *Class* 4, we obtain the following values for K, namely, for T_1,

$$K=\sigma^{\alpha+\gamma}=e^{2\pi i(\alpha+\gamma)/\nu},$$

for S_1,

$$K=\sigma^{\beta+\gamma}=e^{2\pi i(\beta+\gamma)/\nu},$$

for V_1,

$$K=\sigma^{\alpha+\beta}=e^{2\pi i(\alpha+\beta)/\nu}.$$

There are sets of equations for $\alpha+\gamma$, $\beta+\gamma$, and $\alpha+\beta$, similar to those of (6) of the previous section, where l_1, l_2, l_3 denote the l for substitutions T_1, S_1, and V_1 respectively. The value of ν is given by the following equations,

$$(8)\qquad \begin{cases} (a) & \nu=l_1k_1, \\ (b) & \nu=l_2k_2, \\ (c) & \nu=l_3k_3. \end{cases}$$

By the method of the preceding section it may be shown that q must satisfy the congruence $q\equiv\pm1\,(\text{mod. } l_i)$, $(i=1, 2, 3)$, and that l_1, l_2, l_3 may each have only the values $2, 3, 4, 6$. It will not be necessary to examine in this connection for values of ν greater than those in which there may be eight values of q (see theorem IV).

Taking only those combinations of the l's which satisfy the inequality (7) above, we obtain the equation of but one Riemann surface, namely,

$$W^{24} = (z-a)(z-b)^{11}(z-c)^{19}(z-d)^{17},$$

in which $l_1 = 2$, $l_2 = 4$, $l_3 = 6$. It may be shown that all the necessary conditions that the eight parameters may be linearly connected are satisfied. Whether they are actually so connected, we do not need to determine, since the ω-group in this case would be $(0, 3; 2, 4, 6)$ and functions belonging to this group can be expressed more simply by means of hyperelliptic theta functions.* If one of the l's is assumed equal to infinity, nothing new is obtained.

Therefore no new cases are found in Class 5.

Finally, in classes 2 and 3 the value of ν cannot be greater than 6, as otherwise there would be more than one linearly independent ω according to theorem II. All the surfaces for ν equal to 6 or less have been examined and nothing new is found.

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* See HUTCHINSON, *On the functions of the group* $(0, 3; 2, 6, 6)$.

IMPROPER MULTIPLE INTEGRALS

BY

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INTRODUCTION.

This paper is a continuation of a recent article by Professor PIERPONT in this journal.† In that article attention is called to the fact that two methods for defining improper multiple integrals are available;‡ one being that ordinarily followed; the other, that due to DE LA VALLÉE-POUSSIN. In the present paper the theory will be developed from the latter point of view.

It is useful at the outset to introduce a notation that will obviate the difficulty that might arise from confusing the integrals defined by these different methods. Where any doubt is possible we shall denote || by **J** the integral defined by the ordinary process, and by **V** that obtained by following the method of DE LA VALLÉE-POUSSIN. It is evident at a glance that the integral **V** may, in all cases, be proved absolutely convergent. At the time of writing his article, Professor PIERPONT had not been able to prove that his generalized integral **J** must necessarily be absolutely convergent, and hence a more general treatment seemed possible by following the older method of definition. We shall show in § 1 that the integral **J** is absolutely convergent. The discussion in the paper cited above may then be considerably simplified. After the definition in § 2 of the improper multiple integral **V**, it is proved that the existence of either **J** or **V** is a necessary and sufficient condition for the existence of the other, and that both methods lead to the same result. In § 3 various tests for the convergence of singular integrals are outlined.

In this paper we shall regard the field of integration as limited. In a later paper various conditions for continuity, inversion of the order of integration, and reduction of multiple integrals to multiply iterated integrals will be discussed for both finite and infinite fields. In general it will be assumed that, in any partial aggregate in which the integrand is limited, the integral exists.

* Presented to the Society (New York), December 28, 1905. Received for publication February 5, 1906.

† Transactions of the American Mathematical Society, vol. 7 (1906), pp. 153–174.

‡ Loc. cit., p. 153, § 1.

|| The letters **J** and **V** are chosen to suggest respectively the names of the authors, JORDAN and VALLÉE-POUSSIN, who introduced these methods for less general cases.

In nearly all cases, however, the reasoning will hold for an upper or a lower integral. In any instance where the notation is not explained, the reader is referred to the paper cited above, or to a recent treatise* on the theory of functions by the same author.

§ 1. *Absolute convergence of the integrals defined by the first method.*

Before attacking the main problem we must prove some preliminary theorems. When the proof is simple, details are sometimes omitted.

THEOREM 1. *Let f be a limited function of m variables $x_1, x_2, \cdots, x_m$ defined over a limited aggregate $\mathfrak{A}$. Let C be a constant. Then*

$$\bar{\int}_{\mathfrak{A}}(f+C)=\bar{\int}_{\mathfrak{A}}f+\bar{\int}_{\mathfrak{A}}C.$$

This follows at once from the relation $\max(f+C)=\max f+C$ in any cell δ_κ of the division Δ.

THEOREM 2. *Let $f\geqq 0$ be limited and defined over a limited aggregate $\mathfrak{A}$. Let $\mathfrak{B}$ and $\mathfrak{C}$ be partial aggregates of $\mathfrak{A}$, not necessarily unmixed, such that $\mathfrak{A}=\mathfrak{B}+\mathfrak{C}$. Then*

$$\bar{\int}_{\mathfrak{A}}f\leqq\bar{\int}_{\mathfrak{B}}f+\bar{\int}_{\mathfrak{C}}f.$$

For, $\overline{\mathfrak{A}}_\delta\leqq\overline{\mathfrak{B}}_\delta+\overline{\mathfrak{C}}_\delta$. Hence if M_κ is the maximum of f in the cell δ_κ,

$$\sum_{\mathfrak{A}}M_\kappa\delta_\kappa\leqq\sum_{\mathfrak{B}}M_\kappa\delta_\kappa+\sum_{\mathfrak{C}}M_\kappa\delta_\kappa.$$

Passing to the limit $\delta=0$, the theorem is proved.

THEOREM 3. *Let $\mathfrak{B}$ be any limited aggregate, and $\mathfrak{C}$ a partial aggregate of $\mathfrak{B}$. If any improper limiting points of $\mathfrak{C}$ lie in $\mathfrak{B}$, adjoin them to $\mathfrak{C}$ and call the resulting aggregate C. Let $\Delta_1, \Delta_2, \cdots$ be a series of superimposed divisions whose norms $\delta_1, \delta_2, \cdots$ converge to zero. Denote by $\mathfrak{B}_\kappa$ all points of $\mathfrak{B}$ lying in those cells of Δ_κ which contain at least one point of C. Since $\mathfrak{B}_\kappa$ contains C we may write $\mathfrak{B}_\kappa=C+B_\kappa$. Then*

1°. *The aggregates $B_1, B_2, \cdots$ are such that $B_{n-1}\geqq B_n$.*

2°. *The aggregates $B_1, B_2, \cdots$ have no point in common.*

3°. $\operatorname{Lim}_{n=\infty}\bar{B}_n=0$.

To prove 1° note that, since the divisions are superimposed, each aggregate $\mathfrak{B}_{n-1}$ contains at least all of $\mathfrak{B}_n$. As C does not change with the division, $B_{n-1}\geqq B_n$.

Suppose now $B_1, B_2, \cdots$ have a point P in common: P cannot be a limiting point of C, for P lies in B_1 and hence in $\mathfrak{B}_1$ and in $\mathfrak{B}$. But all limiting points

* *Lectures on the Theory of Functions of Real Variables*, vol. 1, Ginn & Co. (1905). This will be referred to as *Lectures*.

of C lying in $\mathfrak{B}$, lie in C and cannot therefore be in any B_κ. Hence for ρ sufficiently small, no point of C and hence no point of $\mathfrak{C}$ lies in $D_\rho(P)$*, the domain of P of norm ρ. Now if ν be taken sufficiently large, every point of that cell of the division Δ_n, $n \geqq \nu$, containing the point P must be entirely within $D_\rho(P)$. But this cell contains at least one point of $\mathfrak{C}$, which is a contradiction. The theorem 2° then follows.

To prove 3°, we assume on the contrary, $\lim_{n=\infty} \overline{B}_n = \beta > 0$. Then $\overline{B}_1 \geqq \overline{B}_2 \geqq \cdots \geqq \beta$; i. e., there are points in common to this series, no matter how great n is taken. This contradicts 2°, and the theorem is proved.

THEOREM 4. *Let f be a limited function defined over a limited aggregate $\mathfrak{A}$. Let $\mathfrak{B}$ be a partial aggregate of $\mathfrak{A}$ such that $\overline{\mathfrak{B}} = \overline{\mathfrak{A}}$. Let f be positive in $\mathfrak{B}$, and negative or zero elsewhere in $\mathfrak{A}$. Then*

$$\overline{\int_{\mathfrak{A}}} f = \overline{\int_{\mathfrak{B}}} f.$$

Every cell containing a point of $\mathfrak{A}$ contains a point of $\mathfrak{B}$, for otherwise $\overline{\mathfrak{B}} < \overline{\mathfrak{A}}$. Hence

$$\overline{\int_{\mathfrak{A}}} f = \lim \sum_{\mathfrak{A}} M_\kappa \delta_\kappa = \lim \sum_{\mathfrak{B}} M_\kappa \delta_\kappa = \overline{\int_{\mathfrak{B}}} f.$$

THEOREM 5. *Let f be limited and defined over a limited aggregate $\mathfrak{A}$. Let $\mathfrak{B}$ denote the points of $\mathfrak{A}$ where $f > 0$. Let $g = f$ in $\mathfrak{B}$ and let $g = 0$ at other points of $\mathfrak{A}$. Then*

$$\overline{\int_{\mathfrak{A}}} g = \overline{\int_{\mathfrak{B}}} f.$$

For let Δ be a division of space of norm δ. Let δ_ι denote cells containing points of $\mathfrak{B}$, while δ_κ denotes cells containing points of $\mathfrak{A}$ but none of $\mathfrak{B}$. Then, since $g_\kappa = 0$,

$$\sum_{\mathfrak{A}} g\delta = \sum g_\iota \delta_\iota + \sum g_\kappa \delta_\kappa = \sum_{\mathfrak{B}} f_\iota \delta_\iota .$$

Passing to limit $\delta = 0$, the theorem is proved.

It is now possible to prove the main theorem, which is a generalization of theorem 21.† There it is proved for relatively measurable fields; here all restrictions on the field are removed.

THEOREM 6. *Let $f(x_1, x_2, \cdots, x_m)$ be defined over any limited aggregate $\mathfrak{A}$. If*

$$\overline{\int_{\mathfrak{A}}} f, \qquad \underline{\int_{\mathfrak{A}}} f$$

are both convergent, they are absolutely convergent.

* For further definition of the symbol, see *Lectures*, p. 153.

† Loc. cit., p. 164.

Let

$$g = f \text{ when } f > 0, \qquad g = 0 \text{ when } f \leqq 0$$

and

$$h = -f \text{ when } f < 0, \quad h = 0 \text{ when } f \geqq 0.$$

Then $|f| = g + h$. The upper integral of $|f|$ will be convergent if the upper integrals of g and h converge. We shall show first that for $\epsilon > 0$ we can find $\delta > 0$, such that, if $\mathfrak{B}$ is an aggregate exterior to the aggregate I of singular points, and $\overline{\mathfrak{B}} < \delta$ then

$$\left| \overline{\int_{\mathfrak{B}}} g \right| < \epsilon. \tag{1}$$

By theorem 5* we can take δ such that, if $\mathfrak{B}_\kappa$ is a partial *unmixed* aggregate of $\mathfrak{B}$,

$$\left| \overline{\int_{\mathfrak{B}_\kappa}} f \right| < \frac{\epsilon}{3}. \tag{2}$$

Denote by $\mathfrak{C}$ the points of $\mathfrak{B}$ where $f > 0$. Adjoin to $\mathfrak{C}$ all its improper limiting points and call the resulting aggregate C. Then† $\bar{C} = \bar{\mathfrak{C}}$. Let $\Delta_1, \Delta_2, \cdots$ be a set of superimposed divisions whose norms converge to zero. Denote by $\mathfrak{B}_\kappa$ all points of $\mathfrak{B}$ in the cells of Δ_κ which contain at least one point of C. Then $\mathfrak{B}_\kappa = C + B_\kappa$. Choose a constant M such that $f + M > 0$ for all points of $\mathfrak{B}$. Then by theorem 1

$$\overline{\int_{\mathfrak{B}_\kappa}} (f + M) = \overline{\int_{\mathfrak{B}_\kappa}} f + \int_{\mathfrak{B}_\kappa} M. \tag{3}$$

Also by theorem 2 and *Lectures*, § 729, 6,

$$\overline{\int_C} (f + M) \leqq \overline{\int_{\mathfrak{B}_\kappa}} (f + M) \leqq \overline{\int_C} (f + M) + \overline{\int_{B_\kappa}} (f + M),$$

or

$$0 \leqq \overline{\int_{\mathfrak{B}_\kappa}} (f + M) - \overline{\int_{\mathfrak{B}}} (f + M) \leqq \overline{\int_{B_\kappa}} (f + M). \tag{4}$$

Similarly

$$0 \leqq \overline{\int_{\mathfrak{B}_\kappa}} M - \overline{\int_C} M \leqq \overline{\int_{B_\kappa}} M. \tag{5}$$

From theorem 3, ν may be chosen so large that for $n \geqq \nu$, $\bar{B}_n$ is less than any assigned quantity. Hence

$$\overline{\int_{B_n}} M < \frac{\epsilon}{3}, \qquad \overline{\int_{B_n}} (f + M) < \frac{\epsilon}{3}. \tag{6}$$

* Loc. cit., p. 159.

† *Lectures*, p. 522, § 715.

Subtracting (4) and (5) and using (6) and (3), we have

$$\left| \bar{\int}_{\mathfrak{P}_n} f - \bar{\int}_C f \right| < \frac{2\epsilon}{3}, \tag{7}$$

and by use of (2),

$$\bar{\int}_C f < \epsilon. \tag{8}$$

But by theorems 4 and 5,

$$\bar{\int}_C f = \bar{\int}_{\mathfrak{G}} f = \bar{\int}_{\mathfrak{P}} g. \tag{9}$$

Finally from (8) and (9),

$$\bar{\int}_{\mathfrak{P}} g < \epsilon.$$

In an analogous manner it is shown that

$$\bar{\int}_{\mathfrak{P}} h < \epsilon.$$

Then

$$\underline{\int}_{\mathfrak{P}} |f| \leqq \bar{\int}_{\mathfrak{P}} |f| = \bar{\int}_{\mathfrak{P}} (g + h) < 2\epsilon.$$

§ 2. *A second method for defining improper multiple integrals. Criteria for convergence.*

The method followed in this new definition has as its foundation an outline given by DE LA VALLÉE-POUSSIN.* Although the increase in the number of variables is an extension, the chief generalization is in regard to the field. While he expressly assumes that the contour in two dimensions is a curve C which is cut but a finite number of times by a line parallel to one of the axes, we put no conditions whatever on the boundary, except to assume that it is of content zero.

Until it has been proved that the integral defined by this method is identical with that discussed in § 1, the symbols **J** and **V** will be used for the integrals defined by the first and second methods; later, this auxiliary notation will be unnecessary.

If f is both positive and negative in $\mathfrak{A}$, we define new functions as follows:

$$\phi = 0 \text{ when } f \leqq 0, \qquad \phi = f \text{ when } f > 0$$

and

$$\psi = 0 \text{ when } f \geqq 0, \qquad \psi = |f| \text{ when } f < 0.$$

*Journal de Mathématiques, ser. 4, vol. 8 (1892), p. 421.

Then $f = \phi - \psi$. Take λ_1, λ_2 arbitrarily large and define the auxiliary functions ϕ_{λ_1}, ψ_{λ_2}.

$$\phi_{\lambda_1} = \phi \text{ for } \phi < \lambda_1, \qquad \phi_{\lambda_1} = \lambda_1 \text{ for } \phi \geqq \lambda_1$$

and

$$\psi_{\lambda_2} = \psi \text{ for } \psi < \lambda_2, \qquad \psi_{\lambda_2} = \lambda_2 \text{ for } \psi \geqq \lambda_2.$$

The corresponding function of f will be denoted by the symbol $f_{\lambda_1 \lambda_2} = \phi_{\lambda_1} - \psi_{\lambda_2}$. For any λ_1 and λ_2, the function $f_{\lambda_1 \lambda_2}$ possesses an integral (definite, upper, or lower) and is a function of $x_1, \cdots, x_m$ and of the parameters λ_1, λ_2. If now

$$\lim_{\lambda_1 = \infty,\ \lambda_2 = \infty} \mathbf{V}_{\mathfrak{A}} f_{\lambda_1 \lambda_2}$$

exists and is finite, we shall define our integral of f as follows:

$$\lim_{\lambda_1,\ \lambda_2 = \infty} \mathbf{V}_{\mathfrak{A}} f_{\lambda_1 \lambda_2} - \mathbf{V}_{\mathfrak{A}} f.*$$

THEOREM 7. *Let $f \geqq 0$ be integrable in $\mathfrak{A}$. Let $\mathfrak{B}$ be a partial aggregate of $\mathfrak{A}$. Then the integral over $\mathfrak{B}$ exists and*

$$\mathbf{V}_{\mathfrak{B}} f \leqq \mathbf{V}_{\mathfrak{A}} f.$$

For any f_λ we know the integral over $\mathfrak{B}$ exists; and

$$\mathbf{V}_{\mathfrak{B}} f_\lambda \leqq \mathbf{V}_{\mathfrak{A}} f_\lambda < \mathbf{V}_{\mathfrak{A}} f.$$

The expression on the left is a monotone increasing function of λ and is limited by the expression on the right which is constant. Hence the integral over $\mathfrak{B}$ converges.

Three other theorems are stated below; the proof of each consists of a simple passage to the limit.

THEOREM 8. *Let $\mathfrak{B}$ and $\mathfrak{C}$ be unmixed partial aggregates of $\mathfrak{A}$, $\mathfrak{B} + \mathfrak{C} = \mathfrak{A}$. If f is integrable in $\mathfrak{B}$ and in $\mathfrak{C}$, then $\mathbf{V}_{\mathfrak{A}} f = \mathbf{V}_{\mathfrak{B}} f + \mathbf{V}_{\mathfrak{C}} f$.*

THEOREM 9. *Let f and g be integrable in $\mathfrak{A}$. Then $f + g$ is integrable in $\mathfrak{A}$ and $\mathbf{V}_{\mathfrak{A}}(f + g) = \mathbf{V}_{\mathfrak{A}} f + \mathbf{V}_{\mathfrak{A}} g$.*

THEOREM 10. *Let $f \geqq 0$, $g \geqq 0$ be two functions integrable in $\mathfrak{A}$. Then $\mathbf{V}_{\mathfrak{A}}(f + g) \geqq \mathbf{V}_{\mathfrak{A}} f \geqq \mathbf{V}_{\mathfrak{A}} f_\lambda$.*

* As an illustration of the methods used in this definition, we may calculate the volume of the solid generated by turning the Witch of Agnesi, $z^2 = 4a^2(2a - y)/y$, about its asymptote as axis. Here $\mathfrak{A}$ is a circle in the xy-plane having its center at the origin. The function which we integrate to find the volume has an infinite discontinuity at the origin. Pass planes parallel to the xy-plane at the distances λ_1 above and λ_2 below. The volume of this truncated solid may be computed by integrating $f_{\lambda_1 \lambda_2}$,

$$\mathbf{V}_{\mathfrak{A}} f_{\lambda_1 \lambda_2} = \frac{8a^4 \pi \lambda_1}{4a^2 + \lambda_1^2} + 4a^3 \pi \tan^{-1} \frac{\lambda_1}{2a} + \frac{8a^4 \pi \lambda_2}{4a^2 + \lambda_2^2} + 4a^3 \pi \tan^{-1} \frac{\lambda_2}{2a}.$$

Passing to the limit for $\lambda_1, \lambda_2 = \infty$, we get $4\pi^2 a^3$ as the volume of the solid.

THEOREM 11. *In order that the integral of f exist, it is necessary and sufficient that the integrals of ϕ and ψ exist.*

With the notation $f_{\lambda_1\lambda_2} = \phi_{\lambda_1} - \psi_{\lambda_2}$,

$$|\mathbf{V}_{\mathfrak{A}} f_{\lambda_1\lambda_2} - \mathbf{V}_{\mathfrak{A}} f_{\lambda_3\lambda_4}| < \frac{\epsilon}{2}, \qquad |\mathbf{V}_{\mathfrak{A}} f_{\lambda_1\lambda_2} - \mathbf{V}_{\mathfrak{A}} f_{\lambda_3\lambda_5}| < \frac{\epsilon}{2}, \qquad (\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5 > R).$$

By subtraction,

$$|\mathbf{V}_{\mathfrak{A}} f_{\lambda_3\lambda_4} - \mathbf{V}_{\mathfrak{A}} f_{\lambda_3\lambda_5}| = |\mathbf{V}\phi_{\lambda_3} - \mathbf{V}\psi_{\lambda_4} - \mathbf{V}\phi_{\lambda_3} + \mathbf{V}\psi_{\lambda_5}| = |\mathbf{V}\psi_{\lambda_4} - \mathbf{V}\psi_{\lambda_5}| < \epsilon.$$

Hence $\mathbf{V}\psi$ exists. Similarly $\mathbf{V}\phi$ exists and the condition is necessary.

To prove the sufficiency of the condition we have, by theorems 9 and 10, $\mathbf{V}(\phi + \psi) = \mathbf{V}\phi + \mathbf{V}\psi \geqq \mathbf{V}\phi_\lambda$. Now from this relation it is evident that $\mathbf{V}\phi_\lambda$ is a limited monotone function. Then $\mathbf{V}\phi$ exists, and in a like manner it is shown that $\mathbf{V}\psi$ exists.

THEOREM 12. *In order that the integral $\mathbf{V}f$ exist, it is necessary and sufficient that $\mathbf{V}|f|$ exist.*

Since $f = \phi - \psi$ and $|f| = \phi + \psi$, this follows at once from the preceding theorem.

THEOREM 13. *In order that $\mathbf{V}_{\mathfrak{A}} f$ be convergent it is necessary and sufficient that, given $\epsilon > 0$, we can find $\delta > 0$, such that $|\mathbf{V}_{\mathfrak{B}} f| < \epsilon$, for $\bar{\mathfrak{B}} < \delta$, where $\mathfrak{B}$ is a partial aggregate of $\mathfrak{A}$.*

It is necessary. For take λ so large that $0 < \mathbf{V}_{\mathfrak{A}}|f| - \mathbf{V}_{\mathfrak{A}}|f|_\lambda < \epsilon/2$ and *a fortiori*

$$0 \leqq \mathbf{V}_{\mathfrak{B}}|f| - \mathbf{V}_{\mathfrak{B}}|f|_\lambda < \frac{\epsilon}{2}. \tag{1}$$

Since now λ is fixed and $|f|_\lambda$ is a limited function, choose $\bar{\mathfrak{B}}$ so small that

$$0 \leqq \mathbf{V}_{\mathfrak{B}}|f|_\lambda < \frac{\epsilon}{2}. \tag{2}$$

Adding (1) and (2)

$$|\mathbf{V}_{\mathfrak{B}} f| \leqq \mathbf{V}_{\mathfrak{B}}|f| < \epsilon.$$

The proof that the condition is sufficient offers no special difficulty.

THEOREM 14. *Let $\mathfrak{B}_1, \mathfrak{B}_2, \cdots$ be partial aggregates of $\mathfrak{A}$ such that $\lim_{n=\infty} \bar{\mathfrak{B}}_n = \bar{\mathfrak{A}}$. If $\mathbf{V}_{\mathfrak{B}_n}|f|$ does not increase indefinitely when n becomes infinite, the integral of f over $\mathfrak{A}$ is absolutely convergent.*

THEOREM 15. *Let $\mathfrak{A}$ be a limited aggregate and $\mathfrak{B}$ a partial aggregate of $\mathfrak{A}$, mixed or unmixed. If the integral of f exists in $\mathfrak{A}$ it exists in $\mathfrak{B}$.*

Since f is integrable in $\mathfrak{A}$ it is absolutely integrable. From theorem 7 it is evident that the integral of $|f|$ exists in $\mathfrak{B}$. Hence f is integrable by theorem 12.*

* The various properties of integrals may be derived with great facility for the generalized integral by using this new definition. In many cases both definitions may be used in the same

Having developed these criteria for the integrals $\mathbf{V}$, the relation to those * defined by Professor PIERPONT may now be investigated.

THEOREM 16. *If the improper multiple integral* $\mathbf{J}$ *exists, then* $\mathbf{V}$ *exists and* $\mathbf{J} = \mathbf{V}$.

Assume first that $f \geqq 0$. Then, by definition, there exists a division of norm δ such that

$$0 \leqq \mathbf{J}_{\mathfrak{A}'_\delta} f = \mathbf{J}_{\mathfrak{A}} f - \mathbf{J}_{\mathfrak{A}_\delta} f < \epsilon.$$

Now f is limited in $\mathfrak{A}_\delta$ and we can take λ so great that $f < \lambda$ in $\mathfrak{A}_\delta$. Then

$$\mathbf{J}_{\mathfrak{A}_\delta} f - \mathbf{V}_{\mathfrak{A}_\delta} f_\lambda = 0, \qquad 0 \leqq \mathbf{J}_{\mathfrak{A}'_\delta} f - \mathbf{V}_{\mathfrak{A}'_\delta} f_\lambda < \epsilon.$$

Adding and using theorem 8,

$$0 \leqq \mathbf{J}_{\mathfrak{A}} f - \mathbf{V}_{\mathfrak{A}} f_\lambda < \epsilon.$$

Hence $\mathbf{V}$ exists and $\mathbf{J} = \mathbf{V}$.

In case $f < 0$, similar reasoning applies. If f is both positive and negative in $\mathfrak{A}$, set $f = \phi - \psi$ as above. Then

$$\mathbf{J}_{\mathfrak{A}} \phi = \mathbf{V}_{\mathfrak{A}} \phi, \qquad \mathbf{J}_{\mathfrak{A}} \psi = \mathbf{V}_{\mathfrak{A}} \psi.$$

Subtracting and using theorem 9, we obtain $\mathbf{J}_{\mathfrak{A}} f = \mathbf{V}_{\mathfrak{A}} f$.

THEOREM 17. *If* $\mathbf{V}$ *exists, then* $\mathbf{J}$ *exists and* $\mathbf{V} = \mathbf{J}$.

Assume as in the preceding theorem, $f \geqq 0$. The other cases present no new difficulty. By definition we can find λ_0 so great that for each $\lambda > \lambda_0$, $\lambda' > \lambda_0$,

$$\mathbf{V}_{\mathfrak{A}} f = \mathbf{V}_{\mathfrak{A}'_\delta} f_\lambda + \mathbf{V}_{\mathfrak{A}_{\delta'}} f_{\lambda'} + \epsilon' \qquad (0 < \epsilon' \leqq \tfrac{1}{2}\epsilon). \tag{1}$$

This relation holds for all divisions Δ since $\mathfrak{A}_\delta$ and $\mathfrak{A}_{\delta'}$ are unmixed aggregates. Fix λ, and take a fixed division Δ of norm δ so small that

$$\mathbf{V}_{\mathfrak{A}'_\delta} < \frac{\epsilon}{2}. \tag{2}$$

Choose further, λ' greater than the maximum of f in $\mathfrak{A}_\delta$. Then

$$\mathbf{V}_{\mathfrak{A}_\delta} f_{\lambda'} = \mathbf{J}_{\mathfrak{A}_\delta} f. \tag{3}$$

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problem to great advantage. In general the proof consists in a simple passage to the limit. Space will permit the statement of a few properties only. 1°. *Let* $f(x) \geqq M$ *in a measurable field* $\mathfrak{A}$. *Then* $\mathbf{V}_{\mathfrak{A}} f > M\mathfrak{A}$. 2°. *Given* $0 < M \leqq f$ *for all points of* $\mathfrak{A}$. *If* f *is integrable, then* $1/f$ *is integrable.* 3°. *Let* f *be integrable in* $\mathfrak{A}$ *and let* g *be integrable and have one sign. If* g *is numerically less than some fixed number* M, *then* $|\mathbf{V}_{\mathfrak{A}} fg| \leqq M |\mathbf{V}_{\mathfrak{A}} f|$. 4°. *Let* f *and* g *be integrable in* $\mathfrak{A}$ *and have no point of infinite discontinuity in common. Then* fg *is integrable.* 5°. *If* $f \geqq g$, *then* $\mathbf{V} f \geqq \mathbf{V} g$. 6°. *Let* f *be integrable in* $\mathfrak{A}$ *and not negative, while* g *is limited and integrable. Let* θ *denote a mean value of* g. *Then* $\mathbf{V}_{\mathfrak{A}} fg = \theta \mathbf{V}_{\mathfrak{A}} f$. 7°. *Let* f *be integrable and not negative. Let* g *be a continuous function of* $x_1, \cdots, x_m$. *Let* $x_1, \cdots, x_m$ *be continuous functions of a parameter* t. *Then* $g = \Phi(t)$ *and* $\mathbf{V} f \cdot g = \Phi(\bar{t})\, \mathbf{V} f$ *where* $\bar{t}$ *is a mean value of* t.

* Loc. cit., p. 158.

From (1), (2), and (3),

$$\mathbf{V}_{\mathfrak{A}} f - \mathbf{J}_{\mathfrak{A}_\delta} f < \epsilon$$

and the theorem is established.

§ 3. *Tests for convergence of singular integrals.*

In many cases, the problem in regard to an integral is solved when it is determined whether, in the neighborhood of certain points, the integral converges. Any method which will obviate the laborious process of a direct integration will have considerable application in practical work. Various tests may be devised for the generalized integral under discussion, some of which are given here.

METHOD I. *If*

$$I_{\lambda_1\lambda_2} = \int_{\mathfrak{A}} f_{\lambda_1\lambda_2}$$

may be developed by Taylor's expansion (in finite form or otherwise)

$$I_{\lambda_1\lambda_2} = \lambda_1^{n_1}[\phi_1(x_1, \cdots, x_m)] + \lambda_1^{n_1-1}[\phi_2(x_1, \cdots, x_m)] + \cdots + \lambda_2^{n_2}[\psi_1(x_1, \cdots, x_m)] + \lambda_2^{n_2-1}[\psi_2(x_1, \cdots, x_m)] + \cdots,$$

where ϕ_i and ψ_i are limited functions, independent of λ_1 and λ_2, the following criterion may at once be written down.

If $n_1 > 0$, or $n_2 > 0$, $I_{\lambda_1\lambda_2}$ diverges;

*if $n_1 \leqq 0$, and $n_2 \leqq 0$, $I_{\lambda_1\lambda_2}$ converges.**

METHOD II. *Let f be regular, except in the neighborhood of an isolated point $\alpha = (\alpha_1, \cdots, \alpha_m)$. Around the point describe an m-dimensional sphere of radius r. On the surface of the sphere suppose that the maximum of the absolute value of the function is M. Suppose further that μ is an exponent which, for all values of r as small as we choose, gives the relation $Mr^\mu < G$, where G is a constant. If now $\mu < m$, the function is integrable. If $Mr^\mu > G$ and $\mu \geqq m$, the function is not integrable.*

Effect a change of variables † by means of the generalized polar transforma-

* In calculating the volume of the solid generated by turning the Witch about its asymptote as axis, § 2, the expression obtained for $I_{\lambda_1\lambda_2}$ may be developed either in finite or infinite form,

$$I_{\lambda_1\lambda_2} = 4a^3\pi^2 + \lambda_1^{-1}(\tfrac{16}{8}a^4\pi) + \lambda_1^{-2}(\tfrac{16}{5}a^5\pi^2) + \cdots + \lambda_2^{-1}(\tfrac{16}{3}a^4\pi) + \lambda_2^{-2}(\tfrac{16}{5}a^5\pi^2) + \cdots.$$

Since $n_1 = 0$, $n_2 = 0$, $I_{\lambda_1\lambda_2}$ converges.

† The equations are

$$x_1 = \rho\cos\theta_1,\ x_2 = \rho\sin\theta_1\cos\theta_2,\ x_3 = \rho\sin\theta_1\sin\theta_2\cos\theta_3,\ \cdots$$

$$x_{m-1} = \rho\sin\theta_1\sin\theta_2\cdots\sin\theta_{m-2}\cos\theta_{m-1},\ x_m = \rho\sin\theta_1\sin\theta_2\cdots\sin\theta_{m-2}\sin\theta_{m-1};$$

where θ_{m-1} varies from 0 to 2π, $\theta_1, \theta_2, \cdots, \theta_{m-2}$ from 0 to π, and ρ from 0 to ∞.

tion. Denote by r and r_1 radii of two concentric spheres about α, and by $\mathfrak{B}$ the aggregate contained in the hollow shell. By a simple computation

$$\int_{\mathfrak{B}} |f| \leqq \frac{8G\left[r^{n-\mu} - r_1^{n-\mu}\right]}{2 \cdot 4 \cdot 6 \cdots n} \cdot \frac{1}{n-\mu}\left(\frac{\pi}{2}\right)^{n/2} \text{ if } n \text{ is even,}$$

$$\leqq \frac{8G\left[r^{n-\mu} - r_1^{n-\mu}\right]}{3 \cdot 5 \cdot 7 \cdots n-2} \cdot \frac{1}{n-\mu}\left(\frac{\pi}{2}\right)^{(n-1)/2} \text{ if } n \text{ is odd.}$$

Suppose now $\mu < m$. Then for $r_1 < r$,

$$\lim_{r=0} \int_{\mathfrak{B}} |f| = 0,$$

and this is a sufficient condition for the convergence of the integral.

To establish the second half of this test it is sufficient to notice that if $n \geqq \mu$ and G_1 is a constant

$$\int_{\mathfrak{B}} |f| > G_1.$$

By theorem 13 the integral diverges.

As a generalization of a test used in integrals of functions of one variable, we get another important test; from another point of view it may be considered as a special case of method II.

METHOD III. *Let $\alpha = (\alpha_1, \cdots, \alpha_m)$ be an isolated point of infinite discontinuity of $f(x_1, \cdots, x_m)$. Let $M > 0$ be a constant and $\mu_1, \cdots, \mu_m$ be a series of exponents each less than unity. If*

$$(|x_1 - \alpha_1|)^{\mu_1}(|x_2 - \alpha_2|)^{\mu_2} \cdots (|x_m - \alpha_m|)^{\mu_m}|f(x)| < M$$

for x in $\mathbf{V}_\delta(\alpha)$ (the vicinity of α of norm δ), then f is integrable.*

YALE UNIVERSITY, NEW HAVEN,
December, 1905.

* For explanation of notation see *Lectures*, § 250, 1.